Functional quantization of rough volatility and applications to the VIX

– Joint work with O. Bonesini and A. Jacquier –

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Outline

A time-line

Quantization was...

- Conceived by Sheppard in **1897** [S1897]
- Studied in the **50's** at Bell Laboratory [GG1992], with applications to Information Theory and Signal Processing
- Established as a Numerical Method to compute (conditional) expectations, in the early **90's** [GL2000]
- Since the **late 1990's** extensions to inf-dim case and **applications**:
  - signal processing and trasmission
  - model-based clustering in Statistics
  - pattern and speech recognition
  - space discretization tool for non-linear problems

Optimal Vector Quantization: visual idea

Approximate a (real-valued) random variable admitting a continuum of values with one valued in a `finite set` of cardinality N:

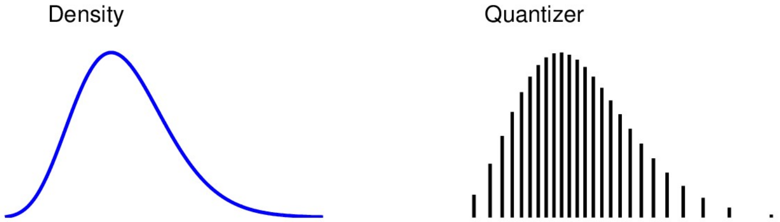


Figure: Picture taken from [MRKP2018].

Optimal Vector Quantization: practice

On $(\Omega, \mathscr{F}, \mathbb{P})$ consider

- X, $\mathbb{R}$-valued random variable
- A N-grid $\Gamma = \{x_1, \dots, x_N\} \in \mathbb{R}$

The quantization of X on Γ is given by the **nearest neighbour projection** of X on Γ:

$$\widehat{X}^{\Gamma} := \mathrm{Proj}_{\Gamma}(X) = \sum_{i=1}^{N} x_i \mathbb{1}_{C_i(\Gamma)}(X),$$

where $(C_i(\Gamma))_{i \in \{1,\dots,N\}}$ is a Voronoi partition of $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$:

$$C_i(\Gamma) \subset \left\{ y \in \mathbb{R} : |y - x_i| = \min_{1 \le j \le N} |y - x_j| \right\} \subset \overline{C_i(\Gamma)}, \quad i = 1, \dots, N.$$

Optimal Vector Quantization: practice

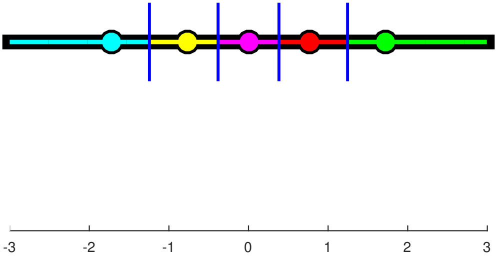


Figure: Example of Voronoi cells associated to a given grid in $\mathbb{R}$.

Optimal Vector Quantization: practice

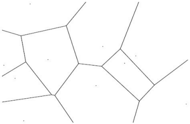


Figure: A 2-dimensional 10-quantizer $\Gamma = \{x_1, \ldots, x_{10}\}$ and its Voronoi diagram. Picture taken from [P2004].

Optimal Vector Quantization: in practice

The grid Γ is chosen in order to minimize the so-called **L^p-mean quantization error** ($p \geq 1$), that is

$$e_{p,N}(X,\Gamma) := \|X - \widehat{X}^{\Gamma}\|_{L^p(\mathbb{P})} = \left(\int_{\mathbb{R}} \min_{1\leq j\leq N} |z - x_j|^p \mu(dz)\right)^{1/p},$$

with μ the distribution of X. In **finite-dimension**, (almost) all the related questions have been investigated and answered. For any $N \in \mathbb{N}$, there exists a unique quadratic optimal N-quantization of $\mathcal{N}(0,1)$.[1]

[1] `http://www.quantize.maths-fi.com/gaussian_database`

Functional Quantization

- Consider a stochastic process $(X_t)_{t\in[0,T]}$ as a random vector taking values in a space of functions $(L^2([0,T],\mathbb{R}^d)$, $\langle f,g\rangle = \int_0^T f(t)g(t)dt)$.
- The **Karhunen-Loève expansion** of the process is crucial!

PROBLEM:

- The K-L expansion of a process is not always available (it is, e.g. for Brownian motion and Brownian bridge).
- "In practice, true optimal quantizers of a process X are out of reach for numerical use..." [PP2005]

SOLUTION: "...but some rate optimal sequences of quantizers do have some semi-closed form." [PP2005]

Product Functional Quantization

Definition

On $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ consider a continuous real-valued centered Gaussian process $X = (X_t)_{t\in[0,T]}$:

$$X_t := \sum_{n=1}^{\infty} \psi_n(t)\xi_n, \quad t \in [0,T], \tag{1}$$

with $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathscr{C}([0,T])$ and $\{\xi_n\}_{n\in\mathbb{N}}$ i.i.d. standard Normal r.v.s.
Then a **product functional quantization** of order N of the process is given by

$$\widehat{X}_t^{\mathbf{d}} := \sum_{n=1}^{m} \psi_n(t)\widehat{\xi}_n^{d(n)} \tag{2}$$

where $\mathbf{d} = (d(1), \ldots, d(m)) \in \mathbb{N}^m$, $\prod_{j=1}^m d(j) \le N$ and, for every $n \in \{1, \ldots, m\}$, $\widehat{\xi}_n^{d(n)}$ is the (unique) optimal quadratic $d(n)$-quantization of the standard Gaussian random variable ξ_n, on the quantizer $\Gamma^{d(n)} = \{x_1, \ldots, x_{d(n)}\}$.

The Riemann-Liouville process

Also know as Type II fractional Brownian motion, or Lévy fractional Brownian motion, it is a Volterra process:

Definition

Consider a unitary time interval $[0,1]$. A centered Gaussian process $Z^H = (Z^H_t)_{t\in[0,1]}$ is a **Riemann - Liouville** process, with Hurst index $H \in (0, \frac{1}{2})$, if:

$$Z^H_t := \int_0^t (t-s)^{H-\frac{1}{2}} dW_s, \qquad t \in [0,1]. \tag{3}$$

- $H = \frac{1}{2} \quad \rightsquigarrow \quad Z^{1/2} = W.$

The RL process for different values of H

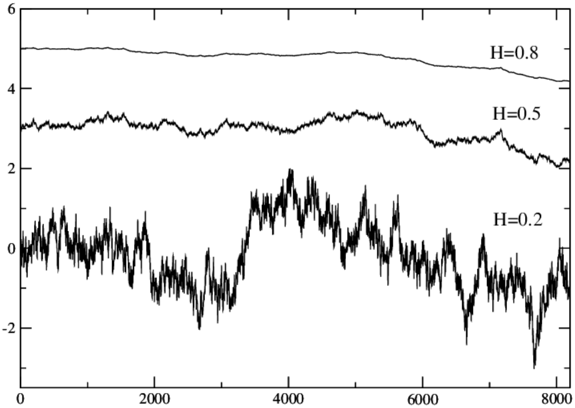


Figure: Picture taken from [V2004].

A series expansion for RL process

A result in [LP2007] guarantees that the following process has the same distribution of a continuous version of the RL process Z^H:

$$Y_t^H := \sum_{n\geq 1} \psi_n^H(t)\xi_n, \qquad t \in [0,1], \tag{4}$$

where

- $\{\xi_n\}_{n\geq 1}$ is a sequence of i.i.d. standard Normal r.v.s,
- $\{\psi_n^H\}_{n\geq 1}$ is a sequence of functions in $\mathscr{C}[0,1]$, given by: $\psi_n^H(t) := \int_0^t (t-s)^{H-\frac{1}{2}} \sqrt{2} \cos\left(\frac{(2n-1)\pi}{2} s\right) ds$, $t \in [0,1]$, $n \in \mathbb{N}$.

The quantization of the RL process

For every $n \geq 1$, discretize ξ_n via $\widehat{\xi}_n^{d(n)} = \mathrm{Proj}_{\Gamma^{d(n)}}(\xi_n)$, over the (unique) optimal grid $\Gamma^{d(n)} = \{x_{i_1}^{d(n)}, \ldots, x_{i_{d(n)}}^{d(n)}\}$ (with $d(n) \geq 1$, $\prod_{n=1}^m d(n) \leq N$), setting:

$$\widehat{Z}_t^{H,\mathbf{d}} = \sum_{n=1}^{m} \psi_n^H(t) \widehat{\xi}_n^{d(n)}.$$

- The corresponding product functional $\mathbf{d}$-quantizer of Z^H is

$$\chi_{\underline{i}}(t) := \sum_{n=1}^{m} \psi_n^H(t)\, x_{i_n}^{d(n)}, \quad \underline{i} = (i_1, \ldots, i_m) \in \prod_{n=1}^{m} \{1, \ldots, d(n)\}. \tag{5}$$

- The probability associated to every trajectory $\chi_{\underline{i}}$:

$$\mathbb{P}(\widehat{Z}^{H,\mathbf{d}} = \chi_{\underline{i}}) = \prod_{n=1}^{m} \mathbb{P}(\xi_n \in C_{i_n}(\Gamma^{d(n)})). \tag{6}$$

Error estimation

Proposition

For any natural number $N \geq 1$*, there exist* $m^*(N) \in \mathbb{N}$ *and* $\mathbf{d}_N^* \in \mathbb{N}^{m^*(N)}$*, with* $\prod_{j=1}^{m} d_N^*(j) \leq N$*, such that*

$$\mathbb{E}\left[\|\widehat{Z}^{H,\mathbf{d}_N^*} - Z^H\|_{L^2[0,1]}^2 \right]^{\frac{1}{2}} \leq K \log(N)^{-H},$$

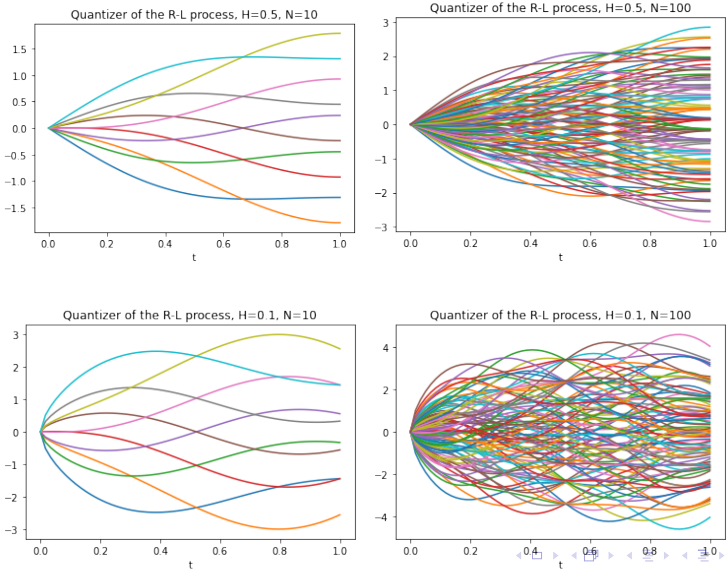
with

$$d_N^*(n) = \left\lfloor N^{\frac{1}{m^*(N)}} n^{-(H+\frac{1}{2})} \left(m^*(N)!\right)^{\frac{2H+1}{2m^*(N)}} \right\rfloor, \qquad n = 1, \dots, m^*(N).$$

Furthermore, we have

$$m^*(N) = \mathcal{O}(\log(N)), \quad \text{as } N \to +\infty.$$

The quantizer for the RL process



Rough Fractional Stochastic Volatility models

- These models are **extensions of classical stochastic volatility models**.
- The volatility is driven by a **rough** process.
- This idea is supported both by empirical:
  - Fukasawa, M.; Takabatake, T. and Westphal, R. ;Is volatility rough?; Mathematical Finance, 2022.
  - Fukasawa, M.; Volatility has to be rough; Quantitative Finance, 2021.
  - Gatheral, J.; Jaisson, T. and Rosenbaum, M.; Volatility is rough; Quantitative Finance, 2018.

  and theoretical studies[2]:
  - Alòs, E.; León, J. A. and Vives J.; On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility; Finance and Stochastics, 2007.
  - Fukasawa, M.;Asymptotic analysis for stochastic volatility: martingale expansion; Finance and Stochastics, 2011.

[2] https://sites.google.com/site/roughvol/home

Rough Fractional Stochastic Volatility models

Equity markets models have to:

- **reproduce** the S&P 500 (SPX) implied volatility surface,
- **calibrate** the VIX Futures.

The rough Bergomi model [BFG2016]

Denote with X the log stock price and with $\mathcal{V}$ the variance process ($v_0(\cdot)$ is the initial forward variance curve, a market input),

$$\begin{cases} X_t = -\frac{1}{2}\int_0^t \mathcal{V}_s ds + \int_0^t \sqrt{\mathcal{V}_s} dB_s, & X_0 = 0, \\ \mathcal{V}_t = v_0(t)\exp\left(2\nu C_H Z_t^H - \frac{(C_H \nu)^2}{H} t^{2H}\right), & \mathcal{V}_0 > 0, \end{cases} \tag{7}$$

where

- $C_H := \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}$, ν positive constants,
- $B := \rho W + \sqrt{1-\rho^2}W^{\perp}$, with $\rho \in [-1,1]$, W and $W^{\perp}$ orthogonal Brownian motions,
- $Z_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ is a Riemann - Liouville process.

VIX

- The **VIX** is the Chicago Board Options Exchange's (CBOE) Volatility Index.
  https://www.cboe.com/tradable_products/vix/

- It is a real-time market index that measures the stock-market's expectation of volatility on a 30-day horizon (Δ).

- It is built from the **Call and Put options on the SPX**.

- The **continuous-time monitoring formula** for the VIX is given by [JMM2018]:

$$VIX_T^2 := \mathbb{E}\left[\frac{1}{\Delta}\int_T^{T+\Delta} d\langle X_s, X_s\rangle \Big| \mathscr{F}_T\right] = \frac{1}{\Delta}\int_T^{T+\Delta} \mathbb{E}\left[\mathcal{V}_s|\mathscr{F}_T\right] ds. \tag{8}$$

VIX Futures

- On March 26, 2004, the CBOE introduced the **VIX Futures**, making the VIX a security.
- Each VIX Future represents the expected implied volatility for the 30 days following the expiration date of the Futures contract itself, that is the forward implied volatility.
- These derivatives are the most liquid derivatives on Equity volatility.

Pricing formula for VIX Futures

The **price of a VIX Future** (maturity T) is computed as in [JMM2018]:

$$
\begin{aligned}
\mathscr{P}_T &:= \mathbb{E}\left[VIX_T \middle| \mathscr{F}_0 \right] \\
&= \mathbb{E}\left[\left(\frac{1}{\Delta} \int_T^{T+\Delta} v_0(t) e^{\left(2\nu C_H Z_t^{H,T,\Delta} + \frac{(\nu C_H)^2}{H} \left((t-T)^{2H} - t^{2H} \right) \right)} dt \right)^{\frac{1}{2}} \middle| \mathscr{F}_0 \right],
\end{aligned} \tag{9}
$$

with $v_0(t)$ as above, and Δ equal to thirty days.

The process to be quantized

The centered Gaussian process $Z^{H,T,\Delta}$ is

$$Z_t^{H,T,\Delta} := \int_0^T (t-s)^{H-\frac{1}{2}} dW_s, \; t \in [T, T+\Delta]. \tag{10}$$

What do we need?

The process to be quantized

The centered Gaussian process $Z^{H,T,\Delta}$ is

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What do we need?

$\leadsto$ series expansion for $Z^{H,T,\Delta}$

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What do we need?

$\rightsquigarrow$ series expansion for $Z^{H,T,\Delta}$ ✓

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The centered Gaussian process $Z^{H,T,\Delta}$ is

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What do we need?

$\rightsquigarrow$ series expansion for $Z^{H,T,\Delta}$ ✓

$\rightsquigarrow$ product functional quantizer for $Z^{H,T,\Delta}$ ✓

... leading to

$$\widehat{\mathrm{VIX}}_T^{\mathbf{d}} := \left(\frac{1}{\Delta} \int_T^{T+\Delta} v_0(t) \exp\left\{ \gamma \widehat{Z}_t^{H,T,\Delta,\mathbf{d}} + \frac{\gamma^2}{2} \left(\int_0^{t-T} K(s)^2 ds - \int_0^t K(s)^2 ds \right) \right\} dt \right)^{\frac{1}{2}}$$

Error bound on the pricing of VIX options

Theorem

Let $F:\mathbb{R}\to\mathbb{R}$ *be a globally Lipschitz-continuous function and* $\mathbf{d}\in\mathbb{N}^m$ *for some* $m\in\mathbb{N}$. *There exists* $\mathfrak{K}>0$ *such that*

$$\left|\mathbb{E}\left[F(VIX_T)\right]-\mathbb{E}\left[F\left(\widehat{VIX}_T^{\mathbf{d}}\right)\right]\right|\leq\mathfrak{K}\,\mathbb{E}\left[\left\|Z^{H,T,\Delta}-\widehat{Z}^{H,T,\Delta,\mathbf{d}}\right\|^2_{L^2[T,T+\Delta]}\right]^{\frac{1}{2}}. \tag{11}$$

Furthermore, for any $N\geq 1$, *there exist* $m_T^*(N)\in\mathbb{N}$ *and* $\mathfrak{C}>0$ *such that, with* $\mathbf{d}^*_{T,N}\in\mathscr{D}^N_{m_T^*(N)}$,

$$\left|\mathbb{E}\left[F(VIX_T)\right]-\mathbb{E}\left[F\left(\widehat{VIX}_T^{\mathbf{d}^*_{T,N}}\right)\right]\right|\leq\mathfrak{C}\log(N)^{-H}. \tag{12}$$

Assumptions

We set the following values for parameters:

- $H = 0.1$,
- $\nu = 1.18778$,

and consider two possible (qualitative) scenarios for the initial forward variance curve v_0:

1. $v_0(t) = (0.234)^2$,
2. $v_0(t) = (0.234)^2(1+t)^2$.

Comparison with MC, Scenario 1

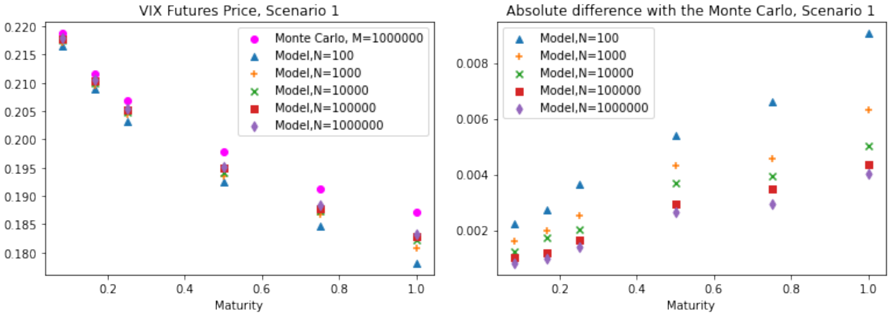

Figure: Approximated price computed with quantization and with Monte-Carlo [JMM2018] as a function of the maturity T, for different numbers of trajectories, in case 1.

Comparison with MC, Scenario 2

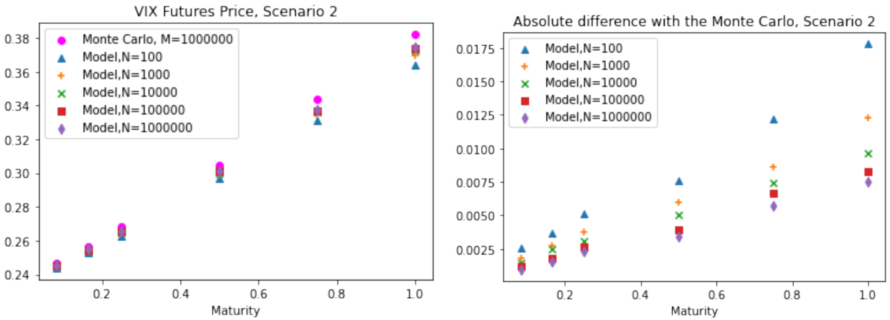

Figure: Approximated price computed with quantization and with Monte-Carlo [JMM2018] as a function of the maturity T, for different numbers of trajectories, in case 2.

Summary

- We have obtained product functional quantizers (and the corresponding errors) for rough Gaussian processes.
- We have applied this to VIX Futures pricing.
- We have an alternative to Monte-Carlo simulations.
- More products priced (as per referees' request!) in the paper: ATM Call on VIX and variance swaps.

- **What about in the future?**
  - Control variate.
  - Calibration.

Thanks for your attention!

The expansion for $Z^{H,T,\Delta}$

$$Z_t^{H,T,\Delta} := \sum_{n=1}^{\infty} \psi_n^{H,T,\Delta}(t)\xi_n, \qquad t \in [T, T+\Delta],$$

where

- $\{\xi_n\}_{n\geq 1}$ is a sequence of i.i.d. standard Normal r.v.s,
- $\{\psi_n^{H,T,\Delta}\}_{n\geq 1}$ is a sequence of functions in $\mathscr{C}[T, T+\Delta]$, given by:

$$\psi_n^{H,T,\Delta}(t) = \frac{\sqrt{2}(T+\Delta)^H}{(n-1/2)^{H+\frac{1}{2}}} \int_0^{\frac{(n-1/2)}{T+\Delta}T} \left(\frac{(n-1/2)}{T+\Delta}t - u\right)^{H-\frac{1}{2}} \cos(\pi u)du$$

$$= \frac{\sqrt{2}(T+\Delta)^H}{(n-1/2)^{H+\frac{1}{2}}} \left\{\cos\left(\frac{(n-1/2)}{T+\Delta}t\pi\right)\left(\zeta_{1/2}\left(\frac{(n-1/2)}{T+\Delta}t, h_1\right) - \zeta_{1/2}\left(\frac{(n-1/2)}{T+\Delta}(t-T), h_1\right)\right)\right.$$

$$\left. + \pi \sin\left(\frac{(n-1/2)}{T+\Delta}t\pi\right)\left(\zeta_{3/2}\left(\frac{(n-1/2)}{T+\Delta}t, h_2\right) - \zeta_{3/2}\left(\frac{(n-1/2)}{T+\Delta}(t-T), h_2\right)\right)\right\}$$

$$\zeta_k(z,h) := \frac{z^{2h}}{2h} {}_1F_2\left(h; k, 1+h; -\frac{1}{4}\pi^2 z^2\right), \qquad \text{for } k \in \left\{\frac{1}{2}, \frac{3}{2}\right\}. \tag{13}$$

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