Estimating the Spot Covariation of Asset Prices – Statistical Theory and Empirical Evidence

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joint work with

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Introduction

Covariance estimation is crucial for

- risk management
- portfolio management
- strategic asset allocation
- asset pricing
- hedging
- quantification of systemic risk
- ...

$\Rightarrow$ Benefit from high-frequency data!

- Recent literature shows strong empirical evidence for distinct time variations in daily and long-term correlations between asset prices.
- But: Surprisingly little is known about intraday variations of asset return covariances.

Questions:

- Do covariances, correlations and betas systematically vary within a day
  $\Rightarrow$ Is there intraday correlation risk?
- How do covariances, correlations and betas behave in extreme market periods?

Why Important?

- Intraday risk management:
  Assess intraday correlation risks.

- Market microstructure research:
  Studies on HF trading, impact of market fragmentation, benefits of circuit breakers.

- Analysis of days with distinct information & "Flash Crashes":
  Asymmetry of correlation behaviour during bull/bear markets at lower frequencies (e.g., De Santis & Gerard, 1997).
  $\Rightarrow$ Similar effects during intraday intervals?

- Crucial for co-jump tests (e.g. Bibinger & Winkelmann, 2014).

In a perfect world ...

- Consider a d-dimensional continuous martingale price process,

$$X_t = X_0 + \int_0^t \Sigma^{1/2}(s)\, dB_s \,, t \in [0,1],$$

where B_t denotes a standard Brownian motion.

- Objects of interest: $\int_0^t \Sigma(s) ds$ and $\Sigma(s)$.

- If X_t is discretely observed with $X_{i/n}, i = 0, \ldots, n$, a natural estimator for $\int_0^t \Sigma(s) ds$ is

$$\mathrm{RC}_n = \sum_{i=1}^{n} (X_{i/n} - X_{(i-1)/n})(X_{i/n} - X_{(i-1)/n})^\top$$

with

$$\mathrm{vec}\left(n^{1/2}\left(\mathrm{RC}_n - \int_0^1 \Sigma(t)\, dt\right)\right) \xrightarrow{\mathcal{L}} N\left(0, \int_0^1 \left(\Sigma(t) \otimes \Sigma(t) dt\right) \mathcal{Z}\right).$$

Example

- For $d = 1$:

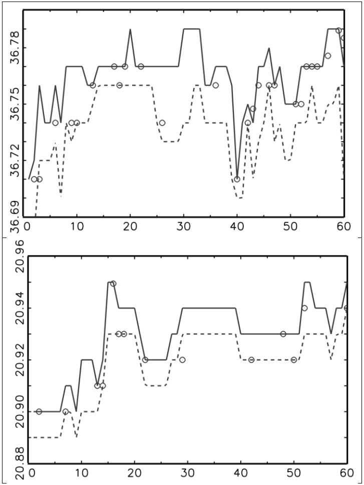
$$n^{1/2}\left(\mathrm{RC}_n - \int_0^1 \sigma^2(s)\,ds\right) \xrightarrow{\mathcal{L}} \mathbf{N}\Big(0, 2\int_0^1 \sigma^4(s)\,ds\Big).$$

- For $d = 2$:

$$\Sigma \otimes \Sigma = \begin{pmatrix} \Sigma_{11}\,\Sigma & \Sigma_{12}\,\Sigma \\ \Sigma_{12}\,\Sigma & \Sigma_{22}\,\Sigma \end{pmatrix}, \ \mathcal{Z} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

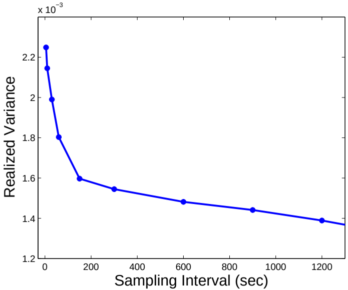
$$\big(\Sigma \otimes \Sigma\big)\mathcal{Z} = \begin{pmatrix} 2\sigma_1^4 & 2\rho\sigma_1^3\sigma_2 & 2\rho\sigma_1^3\sigma_2 & 2\rho^2\sigma_1^2\sigma_2^2 \\ 2\rho\sigma_1^3\sigma_2 & (1+\rho^2)\sigma_1^2\sigma_2^2 & (1+\rho^2)\sigma_1^2\sigma_2^2 & 2\rho\sigma_1\sigma_2^3 \\ 2\rho\sigma_1^3\sigma_2 & (1+\rho^2)\sigma_1^2\sigma_2^2 & (1+\rho^2)\sigma_1^2\sigma_2^2 & 2\rho\sigma_1\sigma_2^3 \\ 2\rho^2\sigma_1^2\sigma_2^2 & 2\rho\sigma_1\sigma_2^3 & 2\rho\sigma_1\sigma_2^3 & 2\sigma_2^4 \end{pmatrix}$$

Real Intraday Price Path

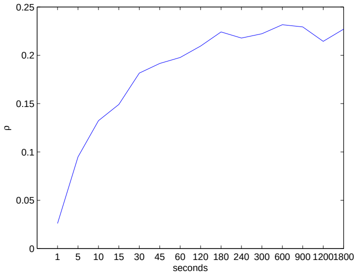


Realized Covariances in Practice

Signature Plot



Epps Effect

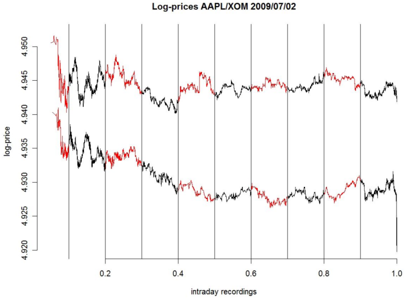


- Challenges:
  - Market microstructure noise
  - Asynchronicity of observations
  - Efficiency
  - Positive definiteness

- Approaches:
  - Hayashi/Yoshida (2011)
  - Realized kernels (Barndorff-Nielsen et al, 2011)
  - Pre-averaging (Christensen et al, 2012)
  - QML (Ait-Sahalia et al, 2010)
  - Spectral estimation (Bibinger/Reiss, 2013)

- Open questions:
  - How to optimally deal with asynchronicity and different speeds in observation frequencies?
  - How to construct spot covariance estimators?

This Paper



Extend and adapt Local Method of Moments (LMM) approach by Bibinger et al. (2014) to spot covariance matrix estimation.
⇒ Build on locally constant approximations of the process
⇒ Robust to microstructure noise and asynchronicity.

- Allow for autocorrelated noise and propose consistent autocorrelation estimators.
  ⇒ Can use tick-by-tick data.
- Derive stable central limit theorem.
  ⇒ Prove rate optimality of estimator.
- Simulation study shows optimal implementation of estimator.
- First empirical evidence on spot covariances & correlations.

Relation to Literature

- Integrated **covariance matrix estimation**:
  - Hayashi/Yoshida (2011);
  - Barndorff-Nielsen et al (2011);
  - Christensen et al (2012);
  - Ait-Sahalia et al (2010);
  - Bibinger et al. (2014).
- **Spot** volatility estimation:
  - Foster & Nelson (1996);
  - Kristensen (2010);
  - Mancini et al. (2012);
  - Bos et al. (2012);
  - Zu & Boswijk (2014).

Outline

1. Introduction
2. LMM: Univariate Case
3. Estimation of Spot Covariances
4. Empirical Results
5. Conclusions

2. Local Method of Moments: Univariate Setting

Univariate Setting

- Consider equi-distantly observed (log) price process:

$$Y_{i/n} = X_{i/n} + \varepsilon_{i/n}, \qquad i = 1, \dots, n, \tag{$\mathcal{E}_0$}$$
$$dX_t = \sigma(t) dB_t, \quad \varepsilon_{i/n} \overset{iid}{\sim} N(0, \eta^2),$$

  where $\varepsilon_{i/n}$ denotes microctructure noise with variance η^2.

- Experiment $(\mathcal{E}_0)$ is asymptotically equivalent to the "continuous-time white noise" process

$$dY_t = X_t dt + \psi dW_t, \tag{$\mathcal{E}_1$}$$

  where $X_t \perp W_t$ and $\psi := \eta/\sqrt{n}$.

- Asymptotic equivalence (in the Le Cam sense) for $n \to \infty$ provided a certain Hölder-regularity of σ_t (Reiss, 2011).

Local Parametric Approximation

- Consider blocks $[kh, (k+1)\,h]\,, k = 0, \ldots, h^{-1} - 1$.
- Assume that block lengths shrink sufficiently fast with increasing n: $h^{\alpha} = o\Big(n^{-1/4}\Big)$ for $\alpha \in (1/2, 1]$.
- Observing ($\mathcal{E}_0$) is asymptotically equivalent to observing

$$dY_t = X_t^h dt + \psi dW_t, \qquad (\mathcal{E}_2)$$

with the efficient (log-) price process

$$dX_t^h = \lfloor \sigma(t) \rfloor_h dB_t, \qquad \lfloor t \rfloor_h = \lfloor t/h \rfloor h,$$

where $\lfloor \sigma(t) \rfloor_h$ denotes the block h-specific constant volatility.

- On block k, we have

$$\tilde{Y}^k_{i^*} = \tilde{X}^k_{i^*} + \varepsilon_{i^*}, \quad i^* = i - khn,$$

with

$$d\tilde{X}^k_{t^*} = \sigma_k \, dB_{t^*}, \quad t^* = t - kh, \quad t \in [kh, (k+1)\, h],$$

σ_k: spot volatility at the beginning of block k.

- Observed returns:

$$\Delta\tilde{Y}^k_{i^*} := \tilde{Y}^k_{i^*} - \tilde{Y}^k_{i^*-1} = \Delta\tilde{X}^k_{i^*} + \varepsilon_{i^*} - \varepsilon_{i^*-1},$$

with $\Delta\tilde{X}^k_{i^*} \overset{\text{i.i.d.}}{\sim} N(0, \sigma_k^2/n)$, $\varepsilon_{i^*} \overset{\text{i.i.d.}}{\sim} N(0, \eta^2)$ and $i^* = 1, \ldots, nh$.

- $\Delta\tilde{Y}^k_{i^*}$ follow MA(1) process with $\mathbb{E}\left[\Delta\tilde{Y}^k_{i^*}\right] = 0$ and

$$\mathbb{C}\text{ov}\left[\Delta\tilde{Y}^k_{i^*}, \Delta\tilde{Y}^k_{i^*-l}\right] = \begin{cases} \sigma_k^2/n + 2\eta^2 & \text{if} \quad l = 0 \\ -\eta^2 & \text{if} \quad l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Spectral Statistics

- Idea: Constructing a statistic in the spectral domain which yields maximal information about $\lfloor \sigma(t) \rfloor_h$.
- Define a set of block-specific functions $\varphi_{jk}(t)$ which form an othornomal system in $L^2([0,1])$.
- Defining $\Phi_{jk}(t) := \int \varphi_{jk}(t)dt$ and setting $\Phi_{jk}(kh) = \Phi_{jk}((k+h)h) = 0$, yields

$$\begin{aligned} \int_{kh}^{(k+1)h} \varphi_{jk}(t)dY_t &= \int_{kh}^{(k+1)h} \varphi_{jk}(t)X_t^h dt + \psi \int_{kh}^{(k+1)h} \varphi_{jk}(t)dW_t, \\ &= -\int_{kh}^{(k+1)h} \Phi_{jk}(t)\lfloor \sigma(t) \rfloor_h dB_t + \psi \int_{kh}^{(k+1)h} \varphi_{jk}(t)dW_t \\ &\overset{d}{=} \left(\int_{kh}^{(k+1)h} \Phi_{jk}^2(t)\lfloor \sigma^2(t) \rfloor_h dt + \psi^2 \right)^{1/2} \xi_{jk}, \end{aligned}$$

where $(\xi_{jk})_{j\geq 1}$ is $N(0,1)$ and independent across j.

- Maximizing information load of $\int_{kh}^{(k+1)h} \varphi_{jk}(t) dY_t$ wrt to $\lfloor \sigma^2(t) \rfloor_h$ yields

$$\varphi_{jk} = \sqrt{2/h} \cos\left[\frac{(t-kh)}{h} j\pi\right] \mathbf{1}_{\{kh,(k+1)h\}}$$

with antiderivative given by

$$\Phi_{jk} = \frac{\sqrt{2h}}{jh} \sin\left[\frac{(t-kh)}{h} j\pi\right] \mathbf{1}_{\{kh,(k+1)h\}}.$$

- Then, for the statistics $S_{jk} = \int_{kh}^{(k+1)h} \varphi_{jk}(t) dY_t$, we have

$$\begin{aligned} S_{jk} &\sim N\left(0, \int_{kh}^{(k+1)h} \Phi_{jk}^2 \lfloor \sigma(t)^2 \rfloor_h dt + \psi^2\right) \\ &= N\left(0, \sigma(kh)^2 \int_{kh}^{(k+1)h} \Phi_{jk}^2 dt + \psi^2\right) \end{aligned}$$

where $\sigma(kh) = \lfloor \sigma(t) \rfloor_k$ for $t \in [kh, (k+1)h]$.

- Thus: $S_{jk} \sim N\left(0, \frac{h^2}{j^2\pi^2} \sigma^2(kh) + \psi^2\right)$

Non-Equidistant Observations

- Consider the process

$$Y_i = X_{t_i} + \varepsilon_i, \qquad (\mathcal{E}_0^*)$$

where $t_i = F^{-1}(i/n)$, where $F : [0,1] \to [0,1]$ is a differentiable cdf with $F'() > 0$ denoting the local observation density.

- Then, $(\mathcal{E}_0^*)$ is asymptotically equivalent to

$$dY_t = X_t dt + \psi(t)\, dW_t, \qquad (\mathcal{E}_1^*)$$

where $\psi(t) := \eta/\sqrt{nF'(t)}$.

- Locally constant approximation:

$$dY_t = X_t^h dt + \lfloor \psi(t) \rfloor_h dW_t, \qquad (\mathcal{E}_2^*)$$

with $\lfloor \psi(t) \rfloor_h = \frac{\eta}{\sqrt{n}} \lfloor \frac{1}{F'(t)} \rfloor_h$.

- Then, under $(\mathcal{E}_2^*)$, we have

$$\int_{kh}^{(k+1)h} \varphi_{jk}(t)dY_t = \int_{kh}^{(k+1)h} \varphi_{jk} X_t^h dt + \int_{kh}^{(k+1)h} \varphi_{jk} \lfloor \psi(t) \rfloor_h dW_t$$
$$\stackrel{d}{=} (||\Phi_{jk}||^2 \sigma(kh)^2 + \psi(kh)^2)^{1/2} \xi_{jk},$$

where $\xi_{jk} \sim N(0,1)$.

- Hence:

$$S_{jk} \sim N\left(0, ||\Phi_{jk}||^2 \sigma(kh)^2 + \frac{\eta^2}{nF'(kh)}\right).$$

with $||\Phi_{jk}||^2 := \int_{kh}^{(k+1)h} \Phi_{jk}^2(t)dt = h^2/j^2\pi^2$.

Local Method of Moments Estimation

- $nh - 1$ independent moment estimators of σ_k^2:

$$\hat{\sigma}_{jk}^2 := \|\Phi_{jk}\|^{-2} \left(S_{jk}^2 - \frac{\eta^2}{nF'(kh)} \right), \qquad j = 1, \ldots, nh - 1.$$

- Combine them to:

$$\hat{\sigma}_k^2 = \sum_{j=1}^{nh-1} w_{jk} \, \hat{\sigma}_{jk}^2 \qquad \text{with} \qquad \sum_{j=1}^{nh-1} w_{jk} = 1.$$

- Minimize variance by choosing weights prop. to Fisher inf. of $\hat{\sigma}_{jk}^2$:

$$w_{jk} = \frac{I_{jk}}{\sum_{l=1}^{nh-1} I_{lk}}, \quad I_{jk} = \frac{1}{2} \left(\sigma_k^2 + \|\Phi_{jk}\|^{-2} \frac{\eta^2}{nF'(kh)} \right)^{-2}.$$

Estimation of Integrated Variance

- Estimator of $\int_0^1 \sigma_t^2 dt$:

$$\widehat{\mathsf{IV}}^{\mathsf{LMM}} := h \sum_{k=0}^{h^{-1}} I_k^{-1} \sum_{j=1}^{nh-1} I_{jk}\, \hat{\sigma}_{jk}^2, \quad I_k := \sum_{j=1}^{nh-1} I_{jk}.$$

- CLT with $n^{1/4}$ rate and $AVAR = 8\eta \int_0^1 \sigma_t^3 dt$ (Reiss, 2011).

3. Estimation of Spot Covariances

Setup

- Efficient log-price X_t follows continuous Itô semi-martingale:

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dB_s, \; t \in [0,1], \tag{1}$$

  where B_s is d-dimensional standard Brownian motion.

- $(d \times d)$ spot covariance matrix: $\Sigma_s = \sigma_s \sigma_s^\top$.

- Observations are non-synchronous and noisy:

$$Y_i^{(p)} = X_{t_i^{(p)}}^{(p)} + \epsilon_i^{(p)} \, , \; i = 0, \ldots, n_p, \; p = 1, \ldots, d \, , \tag{2}$$

  with observation times $t_i^{(p)}$ and observation errors $\epsilon_i^{(p)}$.

- Let $n = \min_{1 \le p \le d} n_p$ denote number of obs. of "slowest" asset.
  $\Rightarrow$ HF asymptotics with $n/n_p \to \nu_p$ for $0 < \nu_p < 1$.

Assumption 1

$(b_s)_{s\in[0,1]}$ *is a càdlàg process with* $b_s \in C^{\nu,R}([0,1],\mathbb{R}^d)$ *for some* $R<\infty$ *and some* $\nu>0$.

Assumption 2

(i) $(\sigma_s)_{s\in[0,1]}$ *follows a càdlàg process with* $\Sigma_s=\sigma_s\sigma_s^\top \geq \underline{\Sigma}$ *uniformly for some strictly positive definite matrix* $\underline{\Sigma}$.

(ii) For $\sigma_s \in C^{\alpha,R}\big([0,1],\mathbb{R}^{d\times d'}\big)$ *with* $R<\infty$ *and* $\alpha\in(0,1/2]$,

$\sigma_s = f\big(\sigma_s^{(1)},\sigma_s^{(2)}\big)$ *with* $f:\mathbb{R}^{2d\times 2d'}\to\mathbb{R}^{d\times d'}$ *continuously differentiable, where*

- $\sigma_s^{(1)}$ *is a continuous Itô semi-martingale and*
- $\sigma_s^{(2)} \in C^{\alpha,R}\big([0,1],\mathbb{R}^{d\times d'}\big)$ *with* $R<\infty$.

(iii) For $\sigma_s \in C^{\alpha,R}\big([0,1],\mathbb{R}^{d\times d'}\big)$ *with* $R<\infty$ *and* $\alpha\in(1/2,1]$, $\sigma^{(1)}$ *vanishes.*

Assumption 3

(i) $\epsilon = \{\epsilon_i^{(p)}, i = 0, \dots, n_p, p = 1, \dots, d\}$ *is independent of* X *and* $\epsilon_i^{(p)}$ *is independent of* $\epsilon_j^{(q)}$ $\forall\ i, j$ *and* $p \neq q$.
(ii) At least first eight moments of $\epsilon_i^{(p)}, i = 0, \dots, n_p$, *exist for* $p = 1, \dots, d$.
(iii) $\mathbb{C}\mathrm{ov}\big(\epsilon_i^{(p)}, \epsilon_{i+u}^{(p)}\big) = 0$ *for* $u > R$, $R < \infty$ *and* $p = 1, \dots, d$.

Define:

$$\eta_p = \eta_0^{(p)} + 2\sum_{u=1}^{R} \eta_u^{(p)}, \quad \textit{with} \quad \eta_u^{(p)} := \mathbb{C}\mathrm{ov}\big(\epsilon_i^{(p)}, \epsilon_{i+u}^{(p)}\big), u \leq R,$$

with $\eta_u^{(p)}, 0 \leq u \leq R$, *constant for all* $0 \leq i \leq n - u$.

Impose $\eta_p > 0$ *for all* p.

Assumption 4

There exist differentiable c.d.f.s F_p, $p = 1, \ldots, d$, *such that observations satisfy* $t_i^{(p)} = F_p^{-1}(i/n_p)$, $0 \leq i \leq n_p, p \in \{1, \ldots, d\}$, *where* $F_p' \in C^{\alpha,R}\big([0,1], [0,1]\big), p = 1, \ldots, d$, *with* α *being the smoothness exponent in Assumption 2 for* $R < \infty$.

Definition 1

In the asymptotic framework with $n/n_p \to \nu_p$, *where* $0 < \nu_p < \infty, p = 1, \ldots, d$, *for* $n \to \infty$, *define the continuous-time noise level matrix*

$$H_s = \operatorname{diag}\left((\eta_p \nu_p (F_p^{-1})'(s))^{1/2}\right)_{1 \leq p \leq d}. \tag{3}$$

Local Method of Moments Estimation

- Estimation using LMM approach by Bibinger et al. (2014).
- Partition interval $[0, 1]$ into blocks $[kh_n, (k+1)h_n], k = 0, \ldots, h_n^{-1} - 1$ with $h_n \to 0$ as $n \to \infty$.
- Approximate original process by process with block-wise constant covariance matrices Σ_{kh_n} and noise levels H_k^n.

$\Rightarrow$ Estimation error can be asymptotically neglected for sufficient smoothness of Σ_t and F_p and block sizes h_n shrinking sufficiently fast.

- Bibinger et al. (2014) propose **integrated** covariance matrix estimator in **simplified** setting.

$\Rightarrow$ Here: estimate **spot** covariance matrix in **generalized** setting.

- Local spectral statistics:

$$S_{jk} = \pi j h_n^{-1} \left(\sum_{i=1}^{n_p} \Big(Y_i^{(p)} - Y_{i-1}^{(p)} \Big) \Phi_{jk} \Big(\frac{t_{i-1}^{(p)} + t_i^{(p)}}{2} \Big) \right)_{1 \le p \le d},$$

where

$$\Phi_{jk}(t) = \frac{\sqrt{2h_n}}{j\pi} \sin\big(j\pi h_n^{-1} (t - kh_n) \big) \mathbf{1}_{[kh_n,(k+1)h_n)}(t), j \ge 1\,.$$

- Can show that

$$\mathrm{Cov}(S_{jk}) = (\Sigma_{kh_n} + \pi^2 j^2 h_n^{-2} \mathbf{H}_k^n)(1 + \mathcal{O}(1)),$$

where $\mathbf{H}_k^n$ has entries

$$\big(\mathbf{H}_k^n\big)^{(pp)} = n_p^{-1} \eta_p (F_p^{-1})'(kh_n)\,,$$

$\Rightarrow$ Estimate Σ_{kh_n} by $S_{jk} S_{jk}^{\top} - \pi^2 j^2 h_n^{-2} \mathbf{H}_k^n$!

An Initial Spot Covariance Matrix Estimator

- Average across frequencies $j = 1, \ldots, J_n^p$ and adjacent blocks:

$$\operatorname{vec}\left(\hat{\Sigma}_{kh_n}^{pre}\right) = (U_{s,n} - L_{s,n} + 1)^{-1} \sum_{k=L_{s,n}}^{U_{s,n}} (J_n^p)^{-1} \sum_{j=1}^{J_n^p} \operatorname{vec}\Big(S_{jk} S_{jk}^\top - \pi^2 j^2 h_n^{-2} \hat{\mathbf{H}}_k^n\Big),$$

where $L_{s,n} = \max\{\lfloor s h_n^{-1} \rfloor - K_n, 0\}$,
$U_{s,n} = \min\{\lfloor s h_n^{-1} \rfloor + K_n, \lceil h_n^{-1} \rceil - 1\}$

- $\hat{\mathbf{H}}_k^n$ is a $\sqrt{n}$-consistent estimator of $\mathbf{H}_k^n$ with diagonal element

$$\left(\hat{\mathbf{H}}_k^n\right)^{(pp)} = \frac{\hat{\eta}_p}{h_n} \sum_{kh_n \le t_i^{(p)} \le (k+1)h_n} \left(t_i^{(p)} - t_{i-1}^{(p)}\right)^2,$$

with $\hat{\eta}_p$ being long-run noise variance estimator.

LMM Spot Covariance Matrix Estimator

- Equal weights for frequencies $j = 1, \ldots, J_n^p$ in general not optimal.
- Increase efficiency: obtain pre-estimated spot covariance matrices using $\operatorname{vec}\left(\hat{\Sigma}_{kh_n}^{pre}\right)$ and derive estimated optimal weight matrices $\hat{W}_j$.

$\Rightarrow$ LMM spot covariance matrix estimator:

$$\operatorname{vec}\left(\hat{\Sigma}_s\right) = \left(U_{s,n} - L_{s,n} + 1\right)^{-1} \sum_{k=L_{s,n}}^{U_{s,n}} \sum_{j=1}^{J_n} \hat{W}_j\left(\hat{\mathbf{H}}_k^n, \hat{\Sigma}_{kh_n}^{pre}\right) \times \operatorname{vec}\left(S_{jk} S_{jk}^\top - \pi^2 j^2 h_n^{-2} \hat{\mathbf{H}}_k^n\right).$$

- Optimal weights proportional to local Fisher info matrices:

$$W_j\big(\mathbf{H}_k^n, \Sigma_{kh_n}\big) = \Big(\sum_{u=1}^{J_n} \big(\Sigma_{kh_n} + \pi^2 u^2 h_n^{-2} \mathbf{H}_k^n\big)^{-\otimes 2}\Big)^{-1} \times \big(\Sigma_{kh_n} + \pi^2 j^2 h_n^{-2} \mathbf{H}_k^n\big)^{-\otimes 2} = I_k^{-1} I_{jk},$$

  with

$$I_{jk} = \big(\Sigma_{kh_n} + \pi^2 j^2 h_n^{-2} \mathbf{H}_k^n\big)^{-\otimes 2},$$

  and $I_k = \sum_{j=1}^{J_n} I_{jk}$.

- Note: $\hat{\Sigma}_s$ symmetric, but not necessarily positive semi-definite.

$\Rightarrow$ E.g., project on space of positive semi-definite matrices.

Pointwise Central Limit Theorem

Theorem 1

Assume a setup with observations of type (2)*, a signal* (1) *and validity of Assumptions 1-4.*

Then, for $h_n = \kappa_1 \log(n) n^{-1/2}$, $K_n = \kappa_2 n^{\beta} (\log(n))^{-1}$ *with constants* κ_1, κ_2 *and* $0 < \beta < \alpha(2\alpha+1)^{-1}$, *for* $J_n \to \infty$ *and* $n/n_p \to \nu_p$ *with* $0 < \nu_p < \infty, p = 1, \ldots, d$, *as* $n \to \infty$, $\hat{\Sigma}_s$ *satisfies:*

$$n^{\beta/2} \operatorname{vec}\left(\hat{\Sigma}_s - \Sigma_s\right) \xrightarrow{d-(st)} \mathbf{N}\Big(0, 2\big(\Sigma \otimes \Sigma_H^{1/2} + \Sigma_H^{1/2} \otimes \Sigma\big)_s \mathcal{Z}\Big), s \in [0,1],$$

where $\Sigma_H = H\big(H^{-1}\Sigma H^{-1}\big)^{1/2} H$ *with noise level* H *from* (3) *and*

$\mathcal{Z} = \mathbb{C}\mathrm{OV}(\operatorname{vec}(ZZ^{\top}))$ *for* $Z \sim \mathbf{N}(0, E_d)$ *being a standard normally distributed random vector.*

Feasible Central Limit Theorem

Corollary 1

Under the assumptions of Theorem 1, $\hat{\Sigma}_s$ satisfies

$$(U_{s,n} - L_{s,n} + 1)^{1/2} \big(\hat{\mathbb{V}}_s^n\big)^{-1/2} \operatorname{vec}\left(\hat{\Sigma}_s - \Sigma_s\right) \xrightarrow{d} \mathbf{N}\Big(0, \mathcal{Z}\Big),\ s \in [0,1]\,,$$

where $$\mathbb{V}_s^n = (U_{s,n} - L_{s,n} + 1)^{-1} \sum_{k=L_{s,n}}^{U_{s,n}} \left(\sum_{j=1}^{J_n} I_{jk}\right)^{-1}.$$

Spot Correlations and Betas

- Spot correlation estimator: $\hat{\rho}_s^{(pq)} = \hat{\Sigma}_s^{(pq)} / \sqrt{\hat{\Sigma}_s^{(pp)} \hat{\Sigma}_s^{(qq)}}$.
- Spot beta estimator: $\hat{\beta}_s^{(pq)} = \hat{\Sigma}_s^{(pq)} / \hat{\Sigma}_s^{(pp)}$.
- Delta method yields:

$$n^{\beta/2} \operatorname{vec}\left(\hat{\rho}_s^{(pq)} - \rho_s^{(pq)}\right) \overset{d-(st)}{\longrightarrow} \mathbf{N}\Big(0, \mathbb{AV}_{\rho,s}\Big),\ s \in [0,1]\,,$$
$$n^{\beta/2} \operatorname{vec}\left(\hat{\beta}_s^{(pq)} - \beta_s^{(pq)}\right) \overset{d-(st)}{\longrightarrow} \mathbf{N}\Big(0, \mathbb{AV}_{\beta,s}\Big),\ s \in [0,1]\,.$$

$\Rightarrow$ Analogously for feasible CLTs.

Estimating Noise Autocovariances

- Estimation of long-run noise variance $\eta_p, p = 1, \ldots, d$, only requires component-wise autocovariance estimates.

  $\Rightarrow$ Restrict analysis to $d = 1$: $n + 1$ observations of $Y_i = X_{t_i} + \epsilon_i, i = 0, \ldots, n$.

- Fix $R \geq 0$ and successively estimate autocovariances by

$$\hat{\eta}_R = (2n)^{-1} \sum_{i=1}^{n} \big(\Delta_i Y\big)^2 + n^{-1} \sum_{r=1}^{R} \sum_{i=1}^{n-r} \Delta_i Y \Delta_{i+r} Y \,,$$

$$\hat{\eta}_r - \hat{\eta}_{r+1} = (2n)^{-1} \sum_{i=1}^{n} \big(\Delta_i Y\big)^2 + n^{-1} \sum_{u=1}^{r} \sum_{i=1}^{n-u} \Delta_i Y \Delta_{i+u} Y \,,$$

$$0 \leq r \leq R - 1 \,.$$

- The variance of $\hat{\eta}_r, 0 \leq r \leq R$, is consistently estimated by

$$\widehat{\mathbb{V}\mathrm{ar}}(\hat{\eta}_r) = n^{-1}\left(V_{r+1}^n + V_r^n + 2C_{r,r+1}^n\right),$$

  with

$$C_{r,r+1}^n = \left(\frac{\hat{\Gamma}_0^{00}}{4} + \frac{1}{2}\sum_{u=1}^{r}\hat{\Gamma}_u^{00} + \sum_{u=0}^{r}\sum_{u'=1}^{r+1}\left(\hat{\Gamma}_0^{uu'} + 2\sum_{q=1}^{R}\hat{\Gamma}_q^{uu'}\right)\right),$$

  and $V_r^n = C_{r,r}^n$, where $\hat{\Gamma}_q^{rr'}$, $q, r, r' \in \{0, \ldots, R\}$ is the fourth sample moment of $\Delta_i Y$.

- In particular, for $r = R$, $\widehat{\mathbb{V}\mathrm{ar}}(\hat{\eta}_R) = n^{-1}V_R^n$.

Theorem 2

Under Assumption 3 and $\mathbb{H}_0^Q : \eta_u = 0$ *for all* $u \geq Q$, $Q = R + 1$, *we have*

$$T_Q^n(Y) = \sqrt{n/V_Q^n}\,\hat{\eta}_Q \xrightarrow{d} \mathbf{N}(0,1)\,.$$

Suitable strategy for selecting R:

- Compute $T_Q^n(Y)$ for $Q \leq \tilde{Q} = \tilde{R} + 1$ "large".
- Incorporate all autocovariances until first hypothesis of zero autocovariance cannot be rejected.

$\Rightarrow$ Using $\hat{R}$, compute long-run noise variance estimate as

$$\hat{\eta} = \hat{\eta}_0 + 2\sum_{u=1}^{\hat{R}} \hat{\eta}_u.$$

4. Empirical Results

Data

- Mid-quotes and transaction prices for 30 most liquid NASDAQ100 constituents and PowerShares QQQ ETF.
- Sample period from May 2010 to April 2014.
- Data sampled from LOBSTER database: `https://lobster.wiwi.hu-berlin.de/`
- Handle (few) errors in the trade and mid-quote samples using cleaning procedures by Barndorff-Nielsen et al. (2009).
- Preliminary analysis: huge share of zero returns in quote data.

$\Rightarrow$ Focus on quote revisions to reduce computational burden.

Choice of Inputs and Implementation

- Theory requires:
  $h_n = \mathcal{O}\big(\log(n) n^{-1/2}\big)$, $J_n = \mathcal{O}\big(\log(n)\big)$,
  J_n^p fixed at a value not "too large" (e.g., $J_n^p = 5$) and $K_n = \mathcal{O}\big(n^{1/4-\varepsilon}\big)$ for $\varepsilon > 0$ "small".
- Introduce proportionality parameters:
  $h_n = \theta_h \log(n) n^{-1/2}$, $J_n = \lfloor \theta_J \log(n) \rfloor$ and $K_n = \lceil \theta_K n^{1/4-\delta} \rceil$, where $\theta_h, \theta_J, \theta_K > 0$.

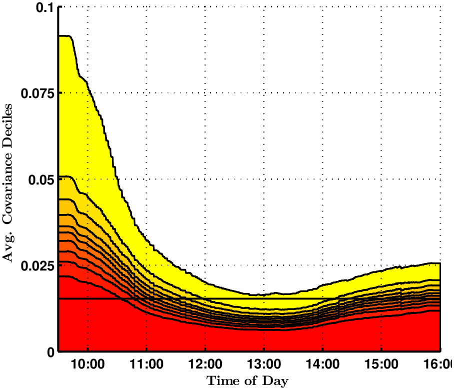
$\Rightarrow$ Based on simulations: $\theta_h = 0.2$, $\theta_J = 8$, $\theta_K = 0.4$, $J_n^p = 5$.

- Estimate
  - 30×30 spot covariance matrices for NASDAQ100 constituents: spot covariances and correlations, volatilities.
  - 31×31 spot covariance matrices including QQQ ETF: spot betas with QQQ as market proxy.

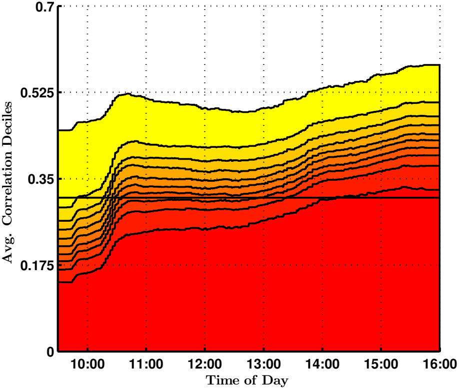
Summary Statistics of Input Values

Input	$q_{0.05}$	Mean	$q_{0.95}$	Std.
$\lceil h_n^{-1} \rceil$	18.000	22.516	29.000	3.922
J_n	48.000	53.532	60.000	3.672
K_n	2.000	2.435	3.000	0.300

Cross-Sectional Deciles of Avg. Covariance and Correlation



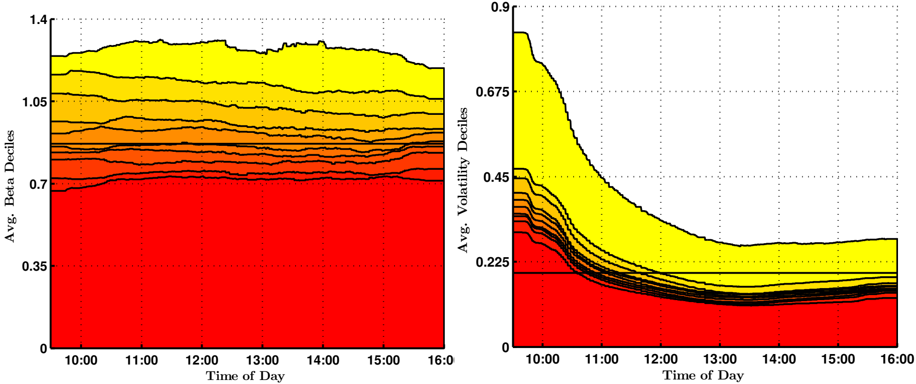
(a) Spot Covariances



(b) Spot Correlations

Spot estimates are averaged across days. Then, cross-sectional sample deciles of across-day averages are computed.

Cross-Sectional Deciles of Avg. Beta and Volatility

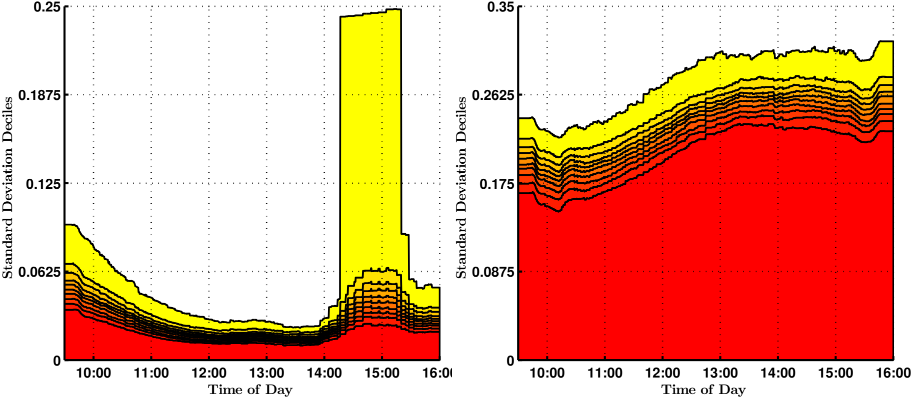


(a) Spot Betas

(b) Spot Volatilities

Spot estimates are averaged across days. Then, cross-sectional sample deciles of across-day averages are computed.

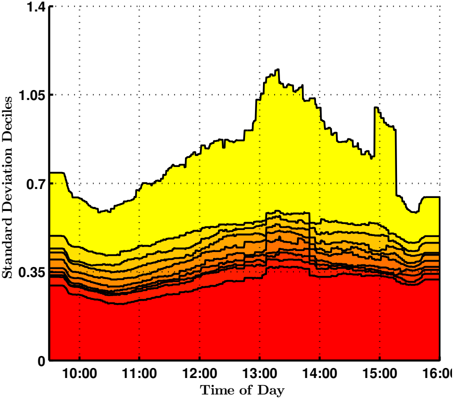
Cross-Sectional Deciles of Std. Dev. of Covariance and Correlation



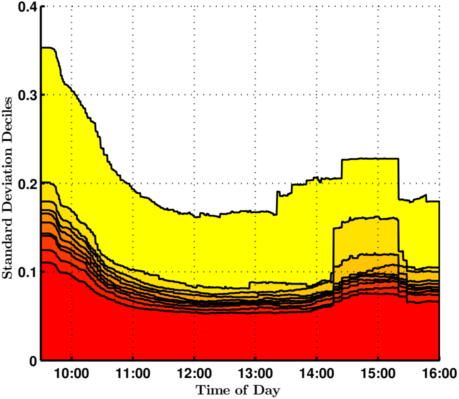
(a) Spot Covariances (b) Spot Correlations

Sample standard deviations of spot estimates are computed across days. Then, cross-sectional sample deciles of across-day standard deviations are computed.

Cross-Sectional Deciles of Std. Dev. of Beta and Volatility



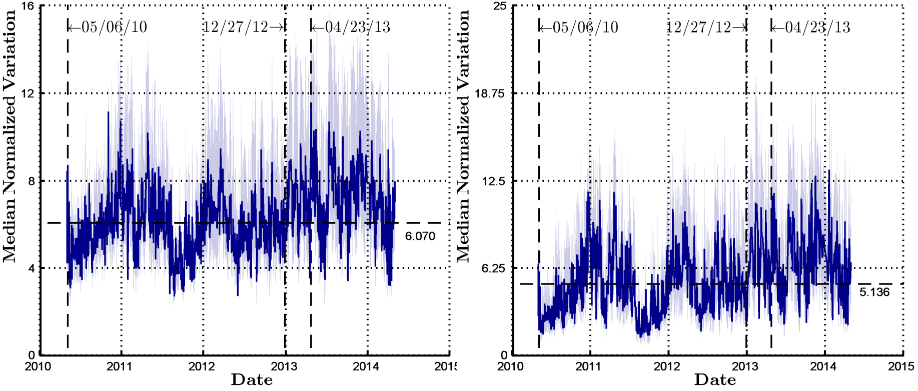
(a) Spot Betas



(b) Spot Volatilities

Sample standard deviations of spot estimates are computed across days. Then, cross-sectional sample deciles of across-day standard deviations are computed.

Cross-Sectional Medians of Intraday Variation Proxy for Covariance and Correlation

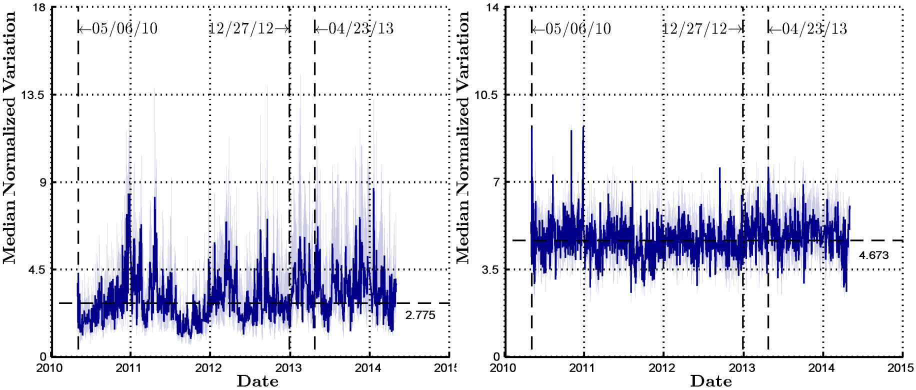


(a) Spot Covariances

(b) Spot Correlations

Total intraday variation proxy: $\sum_{i=1}^{n_g} \left| f\left(t_i\right) - f\left(t_{i-1}\right) \right| \left[\sum_{i=1}^{n_g} \left| f\left(t_i\right) \right| \Delta t_i \right]^{-1}$.

Cross-Sectional Medians of Intraday Variation Proxy for Beta and Volatility



(a) Spot Betas

(b) Spot Volatilities

Total intraday variation proxy: $\sum_{i=1}^{n_g} |f(t_i) - f(t_{i-1})| \left[\sum_{i=1}^{n_g} |f(t_i)| \Delta t_i\right]^{-1}$.

Event I: “Flash Crash” (05/06/10)

(1) Protests in Athens trigger Euro down movement vs. Yen.
U.S. fund managers short-sell E-Mini contracts in vast amounts.

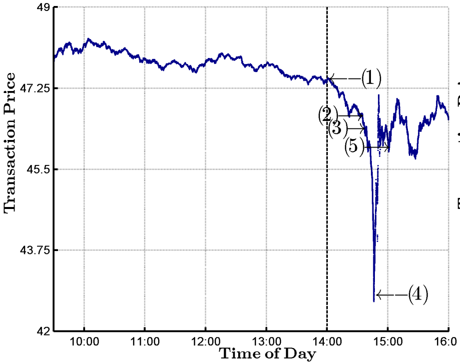
(2) E-Mini market makers cut back trading.

(3) NASDAQ stops order routing to ARCA.

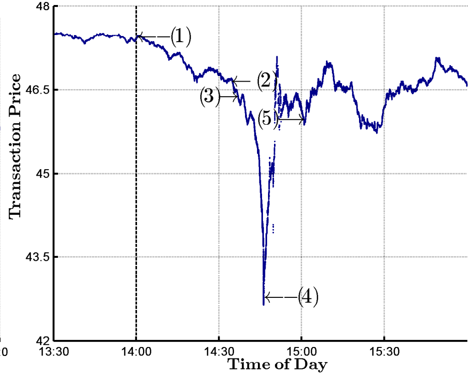
(4) Rumors suggesting that decline occurred due to “fat-finger” error, and not bad news.

(5) NASDAQ resumes routing to ARCA.

05/06/10: QQQ Transaction Prices

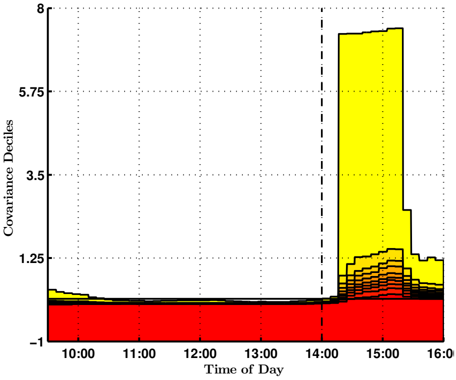


(a) Entire trading day

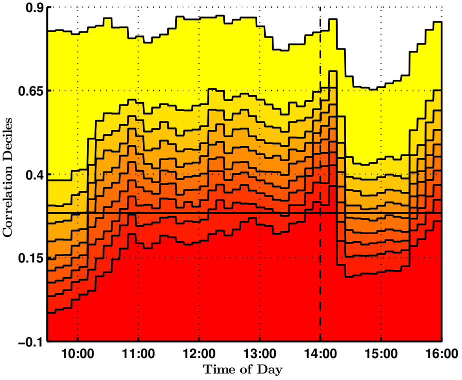


(b) 1:30 pm – 4:00 pm

05/06/10: Cross-Sectional Deciles of Covariance and Correlation

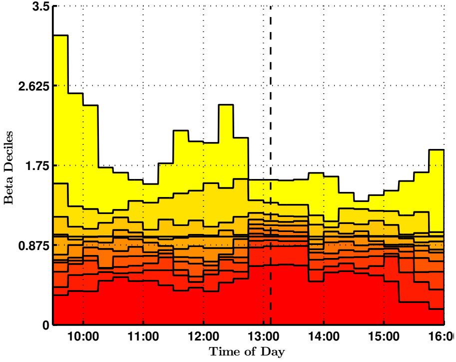


(a) Spot Covariances

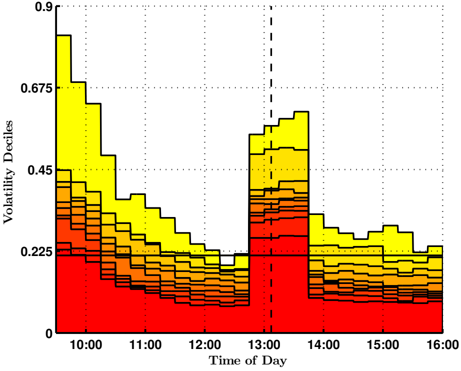


(b) Spot Correlations

05/06/10: Cross-Sectional Deciles of Beta and Volatility



(a) Spot Betas



(b) Spot Volatilities

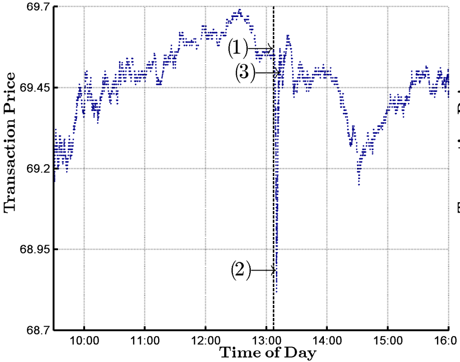
Event II: "Twitter Flash Crash" (04/23/13)

(1) Fake tweet from the account of AP stating "Breaking: Two Explosions in the White House and Barack Obama is injured".

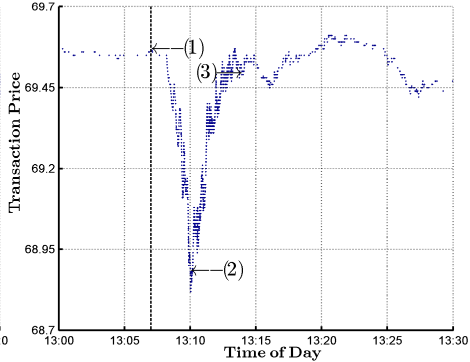
(2) Official denial by AP.

(3) AP's twitter account suspended.

04/23/13: QQQ Transaction Prices

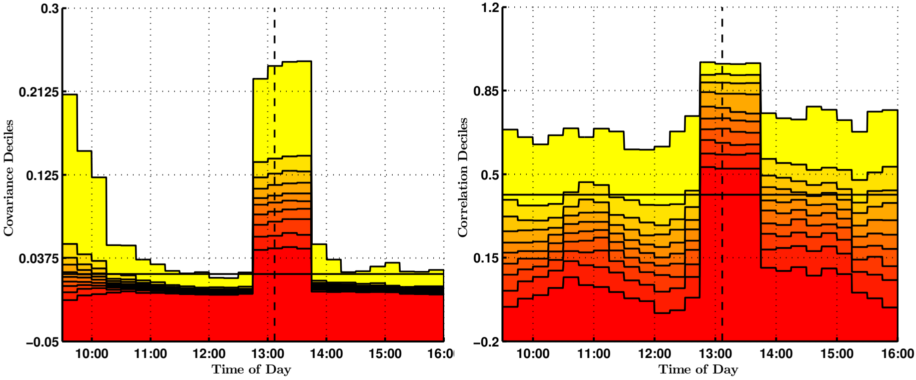


(a) Entire trading day



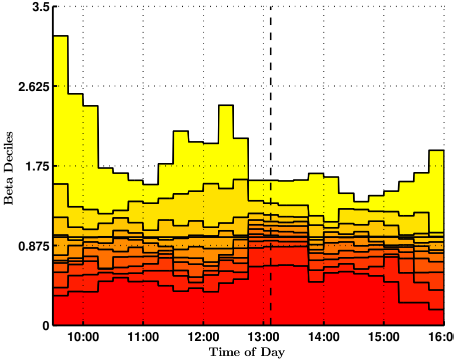
(b) 1:00 pm – 1:30 pm

04/23/13: Cross-Sectional Deciles of Covariance and Correlation

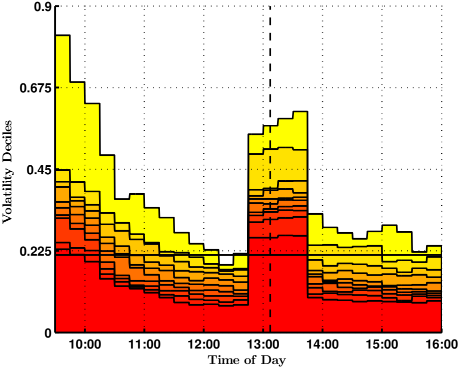


(a) Spot Covariances (b) Spot Correlations

04/23/13: Cross-Sectional Deciles of Beta and Volatility



(a) Spot Betas



(b) Spot Volatilities

5. Conclusions

Conclusions

- Introduce spot covariance matrix estimator relying on LMM approach by Bibinger et al. (2014).
- Extend LMM to allow for autocorrelated noise and provide method for choosing order of dependence.
- Derive stable CLT along with feasible version.
- Simulation study demonstrates how to implement estimator.
- Emprical evidence based on NASDAQ100 stocks:
  - Spot covariances, correlations & volatilities exhibit considerable intraday seasonality.
  - Distinct intraday changes of (co-)volatilities in periods of extreme market movements.