Maxima under Dependence

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Motivation

- Maxima of random variables or vectors are of prime interest in risk management.
- When the variables are i.i.d., asymptotic behavior of maxima is well understood (classical EVT). There is also an extensive literature on the case when the variables form a time series.
- In this talk, the aim is to investigate what happens when the variables are identically distributed but dependent.
- You can think of a large, homogeneous portfolio of claims, in which the claims are dependent, e.g., through some common factor(s).

The central question cast in mathematical terms

Suppose $X_1, X_2, \ldots$ is a sequence of identically distributed univariate random variables (e.g., claims) that are generally not independent.

Define

$$M_n = \max(X_1, \ldots, X_n).$$

Questions to be addressed today:

Under which conditions do there exist sequences of reals (a_n), $a_n > 0$, and (b_n) so that

$$\frac{M_n - b_n}{a_n} \rightsquigarrow H$$

for some non-degenerate df H, and what does H look like?

The i.i.d. case

When $X_1, X_2, \ldots$ are i.i.d., these questions have been long answered by the Fisher–Tippett–Gnedenko Theorem:

If there exist sequences of constants (a_n), $a_n > 0$, and (b_n) so that

$$\frac{M_n - b_n}{a_n} \rightsquigarrow H$$

for some non-degenerate df H, then H must be a generalized extreme-value distribution, given by

$$H_{\xi,\mu,\sigma}(x) = \exp\left\{-\left(1 + \xi\frac{x-\mu}{\sigma}\right)^{-1/\xi}\right\}$$

for all x such that $1 + \xi(x - \mu)/\sigma > 0$.

The time series case: Leadbetter et al. (1983)

Consider a stationary sequence $X_1, X_2, \dots$ with a limited long-range dependence, i.e., so that the so-called $\mathcal{D}(u_n)$ condition holds for a series of suitable thresholds.

If there exist sequences of constants (a_n), $a_n > 0$, and (b_n) so that

$$\frac{M_n - b_n}{a_n} \rightsquigarrow H$$

for some non-degenerate df H, H must be generalized extreme-value.

Let $X_1^*, X_2^*, \dots$ be an i.i.d. sequence with the same marginal distribution and set $M_n^* = \max(X_1^*, \dots, X_n^*)$. Then under regularity conditions

$$\frac{M_n^* - b_n}{a_n} \rightsquigarrow H^* \quad \text{and} \quad \frac{M_n - b_n}{a_n} \rightsquigarrow H$$

where $H = (H^*)^\theta$ for some extremal index $\theta \in (0, 1]$.

Example

Suppose that $X_1, X_2, \ldots$ is an i.i.d. sequence of standard normal variables.

Set the norming constants to be

$$b_n = \Phi^{-1}\left(1 - \frac{1}{n}\right), \quad a_n = \frac{\bar{\Phi}(b_n)}{\varphi(b_n)}$$

where Φ and φ denote the standard normal cdf and density.

Then for all $x \in \mathbb{R}$,

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) \to \Lambda(x) = \exp\{-\exp(-x)\}.$$

Here, Λ denotes the Gumbel extreme-value distribution.

Example cont'd

Let $X_1, X_2, \ldots$ be an stationary sequence of standard normal variables.

Suppose that Berman's condition holds: With $\gamma(n) = \mathrm{cov}(X_1, X_n)$,

$$\lim_{n \to \infty} \gamma(n) \ln(n) = 0.$$

Then as in the i.i.d. case, for all $x \in \mathbb{R}$,

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) \to \Lambda(x) = \exp\{-\exp(-x)\}.$$

Example cont'd: Berman (1962a)

Now consider the sequence $X_1, X_2, \dots$ where for each $n \in \mathbb{N}$,

$$X_n = \sqrt{\varrho} Z_0 + \sqrt{1-\varrho} Z_n;$$

here, $\varrho \in (0,1)$ and $Z_0, Z_1, \dots$ are i.i.d. standard normal variables.

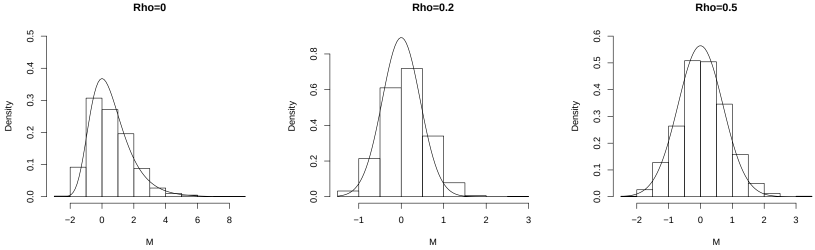
Obviously, $X_1, X_2, \dots$ is a stationary sequence of standard normal variables. However, $\gamma(n) = \varrho$ for all $n \geq 2$, and hence

$$\lim_{n\to\infty} \gamma(n) \ln(n) = \infty.$$

Interestingly, as $n \to \infty$,

$$M_n - \sqrt{1-\varrho} b_n \rightsquigarrow \mathcal{N}(0, \varrho).$$

Illustration



Histograms of the normalized maxima of $n = 10^5$ variables in the i.i.d. case (left), and dependent case with $\varrho = 0.2$ (middle) and $\varrho = 0.5$ (right). Overlaid are the asymptotic densities of the Gumbel (left), and Normal with variance ϱ (middle and right).

Extension to normal variance mixtures

Consider the stationary sequence $X_1, X_2, \ldots$ with

$$X_n = \sqrt{\varrho}\sigma Z_0 + \sqrt{1-\varrho}\sigma Z_n$$

and define, for $n \in \mathbb{N}$,

$$Y_n = \mu + \sqrt{W} X_n,$$

where $\mu \in \mathbb{R}$ and W is a positive random variable independent of (X_i).

$Y_1, Y_2, \ldots$ is then a stationary sequence whose finite-dimensional distributions are elliptical; $W \sim \mathrm{Ig}(\nu/2, \nu/2)$ leads to the t distribution.

A direct calculation shows that if $M_n = \max(Y_1, \ldots, Y_n)$, as $n \to \infty$,

$$\frac{M_n - \mu}{b_n} \rightsquigarrow \sigma\sqrt{1-\rho}\sqrt{W}.$$

Can we do anything at all?

Consider a sequence $X_1, X_2, \dots$ of identically distributed random variables with common distribution F which is assumed to be continuous.

For each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}$, let also

$$F_n(x_1, \dots, x_n) = \Pr(X_1 \le x_1, \dots, X_n \le x_n).$$

From Sklar's Theorem, there exists a unique copula C_n so that

$$F_n(x_1, \dots, x_n) = C_n\{F(x_1), \dots, F(x_n)\}.$$

Using the copula diagonal (Jaworski, 2009) $\delta_n(u) = C_n(u, \dots, u)$, one has

$$\Pr\left(\frac{M_n - b_n}{a_n} \le x\right) = \delta_n\{F(a_n x + b_n)\}.$$

Basic insight

Consider some suitable rate $r(n)$ (typically $r(n) \to \infty$) and write

$$\Pr\left(\frac{M_n - b_n}{a_n} \le x\right) = \delta_n\{F(a_n x + b_n)\}$$
$$= \delta_n\left[\{F^{r(n)}(a_n x + b_n)\}^{1/r(n)}\right]$$

Whether the maximum M_n converges weakly will depend on:

- F, or, in other words, the behavior of the maximum M_n^* of the i.i.d. sequence $X_1^*, X_2^*, \ldots$ with the same common distribution F.
- The behavior of the copula diagonal, notably the limit

$$\lim_{n\to\infty} \delta_n(u^{1/r(n)}), \quad u \in (0,1).$$

First result

Let $X_1, X_2, \ldots$ be a sequence of identically distributed rvs with continuous marginal F, and suppose that the following conditions hold:

(a) *There exist sequences (a_n), $a_n > 0$ and (b_n) such that for all $x \in \mathbb{R}$,*

$$F^n(a_n x + b_n) \to H_{\xi,\mu,\sigma}(x).$$

(b) *There exists a rate function $r : \mathbb{N} \to (0, \infty)$ with $r(n) \to \infty$ as $n \to \infty$ and a continuous function D such that for all $u \in [0, 1]$,*

$$\delta_n\{u^{1/r(n)}\} \to D(u).$$

Then D is in fact a distribution function and for all $x \in \mathbb{R}$,

$$\Pr\left(\frac{M_n - b_{\lceil r(n) \rceil}}{a_{\lceil r(n) \rceil}} \le x\right) \to D\{H_{\xi,\mu,\sigma}(x)\}.$$

Moving maxima example

Consider the moving maximum process: Take a sequence $(Z_i)_{i=-k+1}^{\infty}$ of i.i.d. unit Fréchet variables ($Z_i \sim \Phi_1$) and set

$$X_i = \frac{1}{k+1} \max_{0 \le j \le k} Z_{i-j}$$

This process has limited long-range dependence and its extremal index is $\theta = 1/(k+1)$ (Beirlant et al., 2004). One can also show that

$$C_n(u_1, \ldots, u_n) = \prod_{j=0}^{k-1} \min_{1 \le \ell \le k-j} u_\ell^{1/(k+1)} \min_{n-j \le \ell \le n} u_\ell^{1/(k+1)} \times \prod_{j=1}^{n-k} \min_{j \le \ell \le j+k-1} u_\ell^{1/(k+1)}$$

so that $\delta_n(u) = u^{\eta_n}$, $\eta_n = (k^2 + n)/(k+1)$. Obviously,

$$\lim_{n \to \infty} \frac{\eta_n}{n} = \frac{1}{(k+1)} = \theta.$$

Moving maxima (cont'd)

This means that we can set $r(n) = n$ for each $n \in \mathbb{N}$ and have

$$\delta_n(u^{1/n}) \to u^\theta$$

as $n \to \infty$ for each $u \in [0, 1]$. Consequently,

$$(M_n - a_n)/b_n \rightsquigarrow (\Phi_1)^\theta,$$

where (a_n) and (b_n) are the normalizing sequences from the iid case.

For a continous F in the maximum domain of attraction of a H and

$$Y_i = F^{-1}\{\exp(-1/X_i)\}, \quad i \geq 1,$$

we further have that the maximum $N_n = \max(Y_1, \ldots, Y_n)$ statisfies

$$(N_n - a_n)/b_n \rightsquigarrow H^\theta,$$

where (a_n) and (b_n) are the normalizing sequences from the iid case.

First nasty example

Let $X_1, X_2, \dots$ be a sequence of identically distributed random variables such that

$$X_1 = X_2 = \dots \quad \text{almost surely}$$

and $X_i \sim F$ for some continuous F.

Then for each $n \in \mathbb{N}$,

$$C_n(u_1, \dots, u_n) = \min(u_1, \dots, u_n)$$

is the Fréchet–Hoeffding upper bound and

$$\delta_n(u) = u.$$

This means that for all $n \in \mathbb{N}$, $r(n) = 1$ and in fact

$$M_n \sim F.$$

Second nasty example

From Example 5 in Mai (2018), there exists a sequence of identically distributed random variables such that for all $n \geq 2$,

$$C_n(u_1, \ldots, u_n) = \prod_{i=1}^{n} (1-\theta)^{n-i} u_{(i)}$$

where $u_{(1)} \leq \ldots \leq u_{(n)}$ and $\theta \in (0,1)$.

C_n is the so-called Cuadras–Augé copula and we have

$$\delta_n(u) = u^{(1-(1-\theta)^n)/\theta} = u^{r(n)}$$

with $r(n) = (1-(1-\theta)^n)/\theta$, where $r(n) \to 1/\theta$ as $n \to \infty$.

Clearly, for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\Pr(M_n \leq x) = \delta_n(F(x)) = \{F(x)\}^{r(n)} \to \{F(x)\}^{1/\theta}.$$

There is a limit to what we can do

Let $X_1, X_2, \ldots$ be identically distributed according to a continuous F.

If there exists a function $r\colon \mathbb{N} \to (0, \infty)$ such that

$$r(n) \to \varrho \in (0, \infty)$$

as $n \to \infty$ and for each $u \in [0, 1]$,

$$\delta_n(u^{1/r(n)}) \to D(u)$$

for some continuous D, then for all $x \in \mathbb{R}$, as $n \to \infty$,

$$\Pr(M_n \leq x) \to D(F^{\varrho}(x)).$$

Fisher–Tippett–Gnedenko Theorem: Version I

Let $X_1, X_2, \dots$ be a sequence of identically distributed rvs with continuous marginal F, and suppose that the following conditions hold:

(a) *F is in the maximum domain of attraction of $H_{\xi,\mu,\sigma}$;*

(b) *There exists a rate function $r : \mathbb{N} \to (0, \infty)$ with $r(n) \to \infty$ as $n \to \infty$ and a continuous function D such that for all $u \in [0, 1]$,*

$$\delta_n\{u^{1/r(n)}\} \to D(u).$$

If there exist sequences (b_n) and (a_n) such that $a_n > 0$ for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = G(x),$$

for all continuity points of a non-degenerate G, then there exist $a > 0$ and $b \in \mathbb{R}$ so that $G = D \circ H_{\xi,\tilde{\mu},\tilde{\sigma}}$ where $\tilde{\mu} = (\mu - b)/a$ and $\tilde{\sigma} = \sigma/a$.

Fisher–Tippett–Gnedenko Theorem: Version II

Let $X_1, X_2, \ldots$ be identically distributed according to a continuous F.

Suppose that there exists $r : \mathbb{N} \to (0, \infty)$ and a bijection $\lambda : (0, \infty) \to (0, \infty)$ such that the following conditions hold:

(a) $r(n) \to \infty$ *as* $n \to \infty$ *and for all* $t > 0$, $r(\lceil tn \rceil)/r(n) = \lambda(t)$;

(b) δ_n *is strictly increasing and* $\delta_n\{u^{1/r(n)}\} \to D$ *pointwise for a continuous and strictly increasing* $D : [0, 1] \to [0, 1]$.

If there exist sequences (b_n) and (a_n) such that $a_n > 0$ for all $n \in \mathbb{N}$,

$$\lim_{n \to \infty} \Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = G(x),$$

for all continuity points of a non-degenerate G, then $G = D \circ H$ and H is GEV. If $n/r(n) \to \theta$ for $\theta > 0$, F is in the domain of attraction of H^θ.

Power diagonals

Suppose that for all n, some η_n and all $u \in [0,1]$,

$$\delta_n(u) = u^{\eta_n}.$$

Then if $\eta_n \to \infty$ as $n \to \infty$, we can set $r(n) = \eta_n$.

If F is in the maximum domain of attraction of $H_{\xi,\mu,\sigma}$ with norming constants (a_n), $a_n > 0$ and (b_n),

$$\Pr\left(\frac{M_n - b_{\lceil r(n) \rceil}}{a_{\lceil r(n) \rceil}} \leq x\right) \to H_{\xi,\mu,\sigma}(x).$$

Note also that if $n/\eta_n \to \theta$ for $\theta > 0$ as $n \to \infty$, we get

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) \to H^{1/\theta}_{\xi,\mu,\sigma}(x) = H_{\xi,\mu_\theta,\sigma_\theta}$$

upon setting $r(n) = n$ so that $D(u) = u^{1/\theta}$, $u \in [0,1]$.

A super easy example

Consider an i.i.d. sequence $X_1, X_2, \ldots$.

Clearly, for each $n \in \mathbb{N}$,

$$C_n(u_1, \ldots, u_n) = \Pi(u_1, \ldots, u_n) = u_1 \times \ldots \times u_n.$$

is the independence copula.

Obviously, $\delta_n(u) = u^n$ and we can set $r(n) = n$.

Meta max-stable sequences

Consider a simple max-stable sequence $Z_1, Z_2, \ldots$. This means that Z_i is unit Fréchet and for each $n \in \mathbb{N}$,

$$C_n(u_1, \ldots, u_n) = \exp\{-\ell_n(-\ln u_1, \ldots, -\ln u_n)\}.$$

is an extreme-value copula with stdf ℓ_n. For more flexibility, set

$$X_i = F^{-1}\left\{\exp(-1/Z_i)\right\}.$$

In this case, $\delta_n(u) = u^{\eta_n}$, $\eta_n = \ell_n(1, \ldots, 1)$. For a D-norm construction

$$\ell_n(x_1, \ldots, x_n) = \mathrm{E}\{\max_{1 \leq i \leq n}(x_i W_i)\}$$

where $W_1, W_2, \ldots$ is a sequence of positive rvs. with unit mean, η_n is the extremal coefficient (Smith, 1990; Falk, 2019).

Meta max-stable sequences (cont'd)

Our theory applies as soon as

$$\lim_{n\to\infty} \ell_n(1,\dots,1) = \infty.$$

A classical example where this works is the logistic stdf with

$$\ell_n(x_1,\dots,x_n) = (|x_1|^\theta + \ldots + |x_n|^\theta)^{1/\theta}$$

for $\theta \geq 1$. Here,

$$\eta_n = \ell_n(1,\dots,1) = n^{1/\theta}.$$

However, the Cuadras–Augé copula is also extreme-value, and yet

$$\ell_n(1,\dots,1) = (1-(1-\theta)^n)/\theta \to 1/\theta.$$

Characterizing transformations leading to GEV limits

Suppose that $D\colon [0,1] \to [0,1]$ is a continuous and strictly increasing distribution function. Then

$$D \circ H$$

is GEV for all GEV H if and only if there exist $\alpha > 0$ and $c > 0$ so that

$$D(u) = \exp\{-\alpha(-\ln u)^c\}$$

for all $u \in [0,1]$.

Convergence of the copula diagonal is necessary

Let $X_1, X_2, \ldots$ be a sequence of identically distributed rvs with continuous marginal F, and suppose that the following conditions hold:

(a) *There exist sequences (a_n), $a_n > 0$ and (b_n) such that for all $x \in \mathbb{R}$,*

$$F^n(a_n x + b_n) \to H_{\xi,\mu,\sigma}(x).$$

(b) *There exists a rate function $r : \mathbb{N} \to (0, \infty)$ with $r(n) \to \infty$ as $n \to \infty$ and $1/r(n) = O(1/n)$, and a distribution function D on $[0, 1]$ so that*

$$\Pr\left(\frac{M_n - b_{\lceil r(n) \rceil}}{a_{\lceil r(n) \rceil}} \leq x \right) \to D\{H_{\xi,\mu,\sigma}(x)\}.$$

for all continuity points x of $D \circ H$,

Then for all continuity points $u \in [0, 1]$ of D,

$$\delta_n\left\{u^{1/r(n)}\right\} \to D(u).$$

Example

Consider a Gaussian AR(1) process

$$X_n = \phi X_{n-1} + Z_n,$$

with $X_0 = 0$, $\phi \in [0, 1)$ and iid $Z_n \sim \mathcal{N}(0, \sigma^2)$. Here, C_n is Gaussian with

$$\delta_n(u) = \Phi_{\Sigma_n}\left(\sigma\Phi^{-1}(u)/\sqrt{1-\phi^2}, \ldots, \sigma\Phi^{-1}(u)/\sqrt{1-\phi^2}\right)$$

where $(\Sigma_n)_{ij} = \phi^{|i-j|}\sigma^2/(1-\phi^2)$. Because $\ln(n)\mathrm{cov}(X_1, X_n) \to 0$,

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) \to \Lambda(x).$$

where (a_n), $a_n > 0$ and (b_n) are the normalizing sequences of the corresponding iid series. This means that for all $u \in [0, 1]$,

$$\delta_n\{u^{1/n}\} \to u.$$

Time series with limited long-range dependence

Suppose that $X_1, X_2, \dots$ is a stationary sequence and $X_1^*, X_2^*, \dots$ iid with the same (marginal) distribution F.

Set $M_n = \max(X_1, \dots, X_n)$ and $M_n^* = \max(X_1^*, \dots, X_n^*)$. If

$$\frac{M_n^* - b_n}{a_n} \rightsquigarrow H$$

and

$$\frac{M_n - b_n}{a_n} \rightsquigarrow H^\theta$$

for some $\theta \in (0, 1]$, then necessarily

$$\delta_n(u^{1/n}) \to u^\theta$$

as $n \to \infty$.

Archimax diagonals

Take a simple max-stable sequence $Z_1, Z_2, \ldots$, an independent positive rv. V with Laplace transform ψ, and set

$$Y_i = VZ_i.$$

We can obtain a sequence $X_1, X_2, \ldots$ with common distribution F, viz.

$$X_i = F^{-1}(\psi(1/Y_i))$$

C_n is then an Archimax copula, i.e., for all n and $u_1, \ldots, u_n \in [0, 1]$,

$$C_n(u_1, \ldots, u_n) = \psi[\ell_n\{\psi^{-1}(u_1), \ldots, \psi^{-1}(u_n)\}];$$

where ψ is a completely monotone Archimedean generator and ℓ_n an stdf. In particular, if $Z_1, Z_2, \ldots$ are i.i.d., C_n is an Archimedean copula.

Archimax diagonals (cont'd)

The diagonal of an Archimax copula C_n has the form

$$\delta_n(u) = \psi\{\eta_n \psi^{-1}(u)\}, \quad \eta_n = \ell_n(1, \ldots, 1).$$

If $\eta_n \to \infty$ as $n \to \infty$ and $1 - \psi(1/\cdot) \in \mathrm{RV}_{-\rho}$, then

$$r(n) = \left\lfloor \frac{1}{1 - \psi(1/\eta_n)} \right\rfloor, \quad D(u) = \psi[\{-\ln(u)\}^{1/\rho}]$$

so that

$$\Pr\left(\frac{M_n - b_{\lceil r(n) \rceil}}{a_{\lceil r(n) \rceil}} \leq x \right) \to \psi[\{-\ln H_{\xi,\mu,\sigma}(x)\}^{1/\rho}]$$

provided that F is in the maximum domain of attraction of $H_{\xi,\mu,\sigma}$ with normalizing constants (a_n), $a_n > 0$ and (b_n).

A bit more on Archimax diagonals

- It is worth noting that

$$\psi[\{-\ln H_{\xi,\mu,\sigma}(x)\}^{1/\rho}] = \psi[-\ln H_{\rho\xi,\mu,\rho\sigma}(x)].$$

- When $\ell_n(x_1, \ldots, x_n) = x_1 + \ldots + x_n$, $X_1, X_2, \ldots$ is an exchangeable sequence and C_n is Archimedean. One then recovers the results of Berman (1962b), Ballerini (1994), and Wüthrich (2004).

- When $-\psi'(0) \in (0, \infty)$ and $\eta_n/n \to \theta$, we get that

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) \to \psi\left\{-\ln H_{\xi,\mu,\sigma}^{-\theta/\psi'(0)}(x)\right\}$$

where (a_n), $a_n > 0$ and (b_n) are the constants from the iid case.

Illustration of D for various ψ

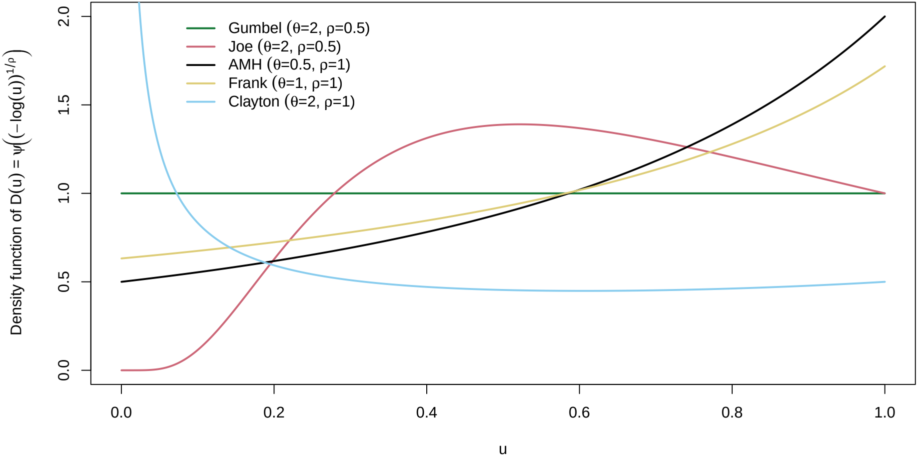


Illustration when ψ is Clayton and $\xi > 0$

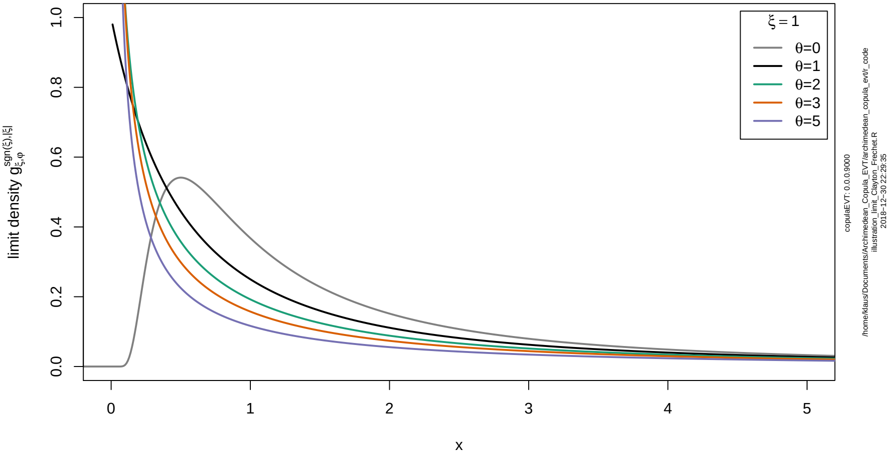


Illustration when ψ is Clayton and $\xi = 0$

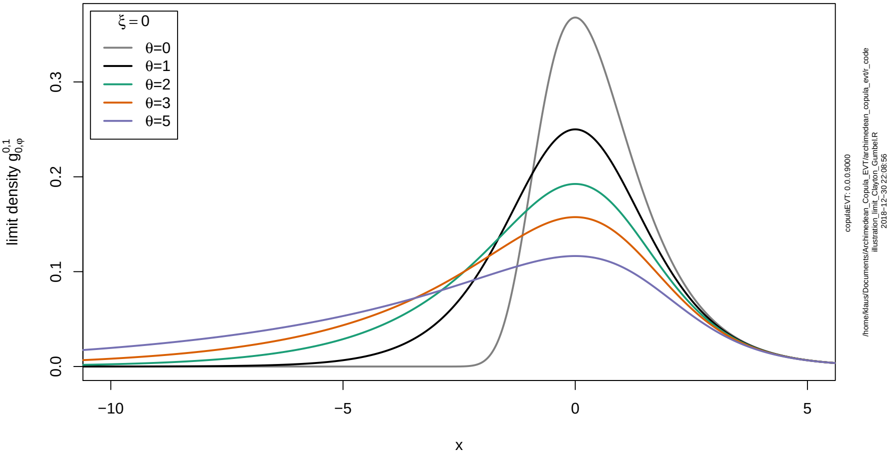
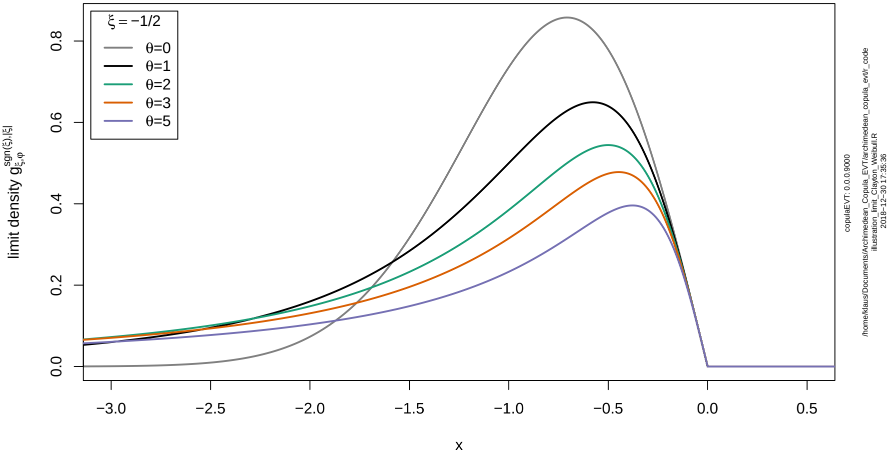


Illustration when ψ is Clayton and $\xi < 0$



Convergence rates

Consider a sequence $X_1, X_2, \dots$ of identically distributed random variables with continuous common distribution F such that

$$\sup_{x\in\mathbb{R}} |F^n(a_n x + b_n) - H(x)| \le \beta(n),$$

for some EVD H. Suppose also that there eixsts $r\colon \mathbb{N} \to (0,\infty)$ so that $r(n) \to \infty$ as $n \to \infty$ and

$$\sup_{u\in[0,1]} |\delta_n(u^{1/r(n)}) - D(u)| \le s(n)$$

If D is Hölder continuous with constant K and parameter $0 < \kappa \le 1$, then

$$\sup_{x\in\mathbb{R}} \left| \Pr\left(\frac{M_n - b_{\lceil r(n) \rceil}}{a_{\lceil r(n) \rceil}} \le x \right) - D \circ H(x) \right| \le K \left(\beta\left(\lceil r(n) \rceil\right) + 3e^{-1}/r(n) \right)^{\kappa} + s(n)$$

Moving maxima one last time

Consider $X_1, X_2, \ldots,$ such that $X_i \sim \mathcal{N}(0,1)$ and the dependence of the moving maxima process. We saw that for $r(n) = n$ and $u \in [0,1]$,

$$\delta_n(u^{1/r(n)}) = \delta_n(u^{1/n}) = u^{(k+n)/(n(k+1))} \to u^{1/(k+1)}.$$

The limit is Hölder continuous with $K = 1$ and $\kappa = 1/(k+1)$. Also,

$$\sup_{u \in [0,1]} |\delta_n(u^{1/r(n)}) - D(u)| = \frac{k}{n+k}\left(1 + \frac{k}{n}\right)^{-n/k}.$$

Hall (1979) provides sequences of constants so that

$$\sup_{x \in \mathbb{R}} |\Phi^n(a_n x + b_n) - \Lambda(x)| \leq 3/\ln(n).$$

The previous result shows that

$$\sup_{x \in \mathbb{R}} \left| \Pr(M_n \leq a_n x + b_n) - \Lambda(x)^{1/(k+1)} \right| \leq \frac{k}{n+k}\left(1 + \frac{k}{n}\right)^{-n/k} + \left(\frac{3}{\ln(n)}\right)^{1/(k+1)}$$

Outlook

- Better understand the constraints on C_n (and its diagonal) when C_n is extendible.
- The development of inferential tools based on these results.
- Generalizations to the multivariate case.

Thank you for your attention!

NSERC CRSNG

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