Convergence order of Euler-type schemes for SDEs in dependence of the Sobolev regularity of the drift

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Object of study

Consider the following SDE on the $\mathbb{R}^d$:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \qquad X_0 = \xi,$$

where $\xi \in \mathbb{R}^d$, $\mu : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, B is d-dimensional Brownian motion.

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Consider the following SDE on the $\mathbb{R}^d$:

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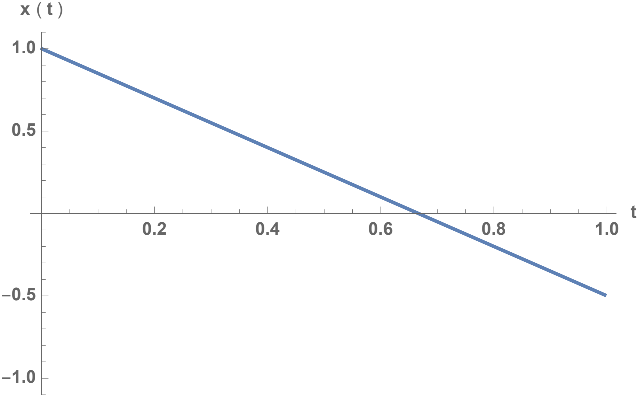
$\mu \equiv -1.5$



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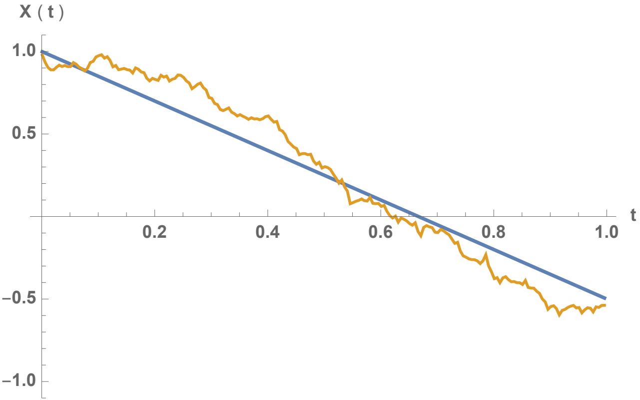
$\mu \equiv -1.5$, $\sigma \equiv 1$



The Euler-Maruyama method

Algorithm

- *Choose a time grid* $0 = t_0 < t_1 < \cdots < t_n = T$.
- *Start at time 0:* $\hat{X}_{t_0} = \xi$
- *Now*

$$\hat{X}_{t_{k+1}} = \hat{X}_{t_k} + \mu(\hat{X}_{t_k}) \cdot (t_{k+1} - t_k) + \sigma(\hat{X}_{t_k}) \cdot \Delta_{k+1}$$

where $\Delta_{k+1} \sim \mathcal{N}(0, t_{k+1} - t_k)$.

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Does this method work?

Theorem

If μ *and* σ *are Lipschitz continuous, then the Euler-Maruyama method has strong convergence order* $1/2$*, that is*

$$\left(\mathbb{E}\left[\|X_t - \hat{X}_t\|^2\right]\right)^{1/2} \leq c \cdot (\max\{t_{k+1} - t_k\})^{1/2}.$$

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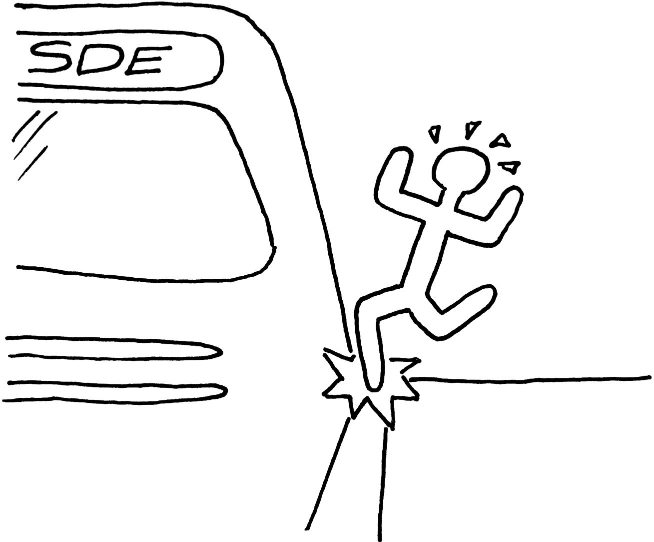

Two examples from maths for finance and economics

1. The dividend paying firm (S. 2016):
   Dividend payments $\alpha(X_t) = \mathbf{1}_{\{X_t \geq \text{ threshold}\}}$

$$dX_t = (m - \alpha(X_t))dt + \sigma dB_t$$

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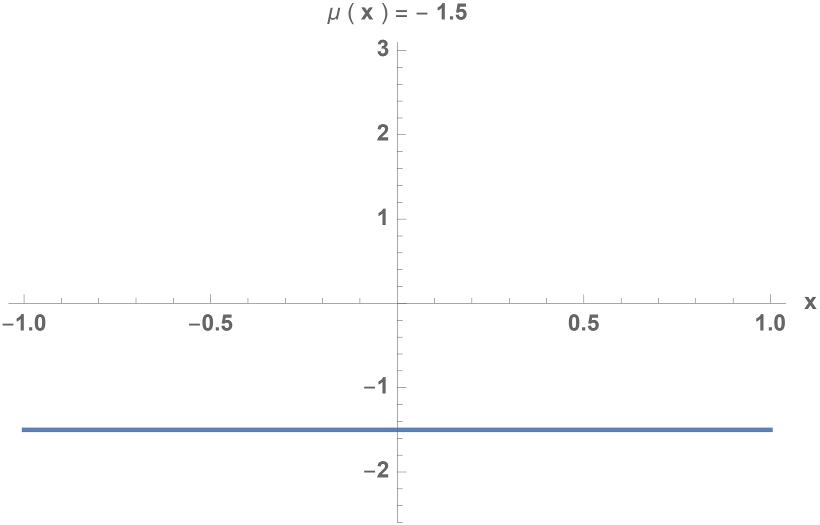
2. The energy storage manager (Shardin, S. 2016):

$$dS_t = (m - S_t)dt + \sigma dB_t$$
$$dF_t = \left(\mathbf{1}_{\{S_t \leq \text{ threshold}_1(S_t, F_t)\}} - \mathbf{1}_{\{S_t \geq \text{threshold}_2(S_t, F_t)\}}\right) dt$$

Can discontinuities matter?



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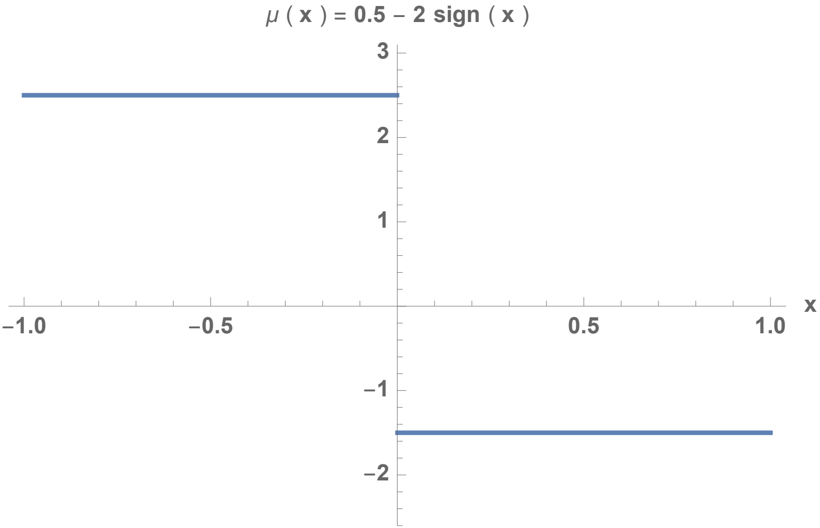

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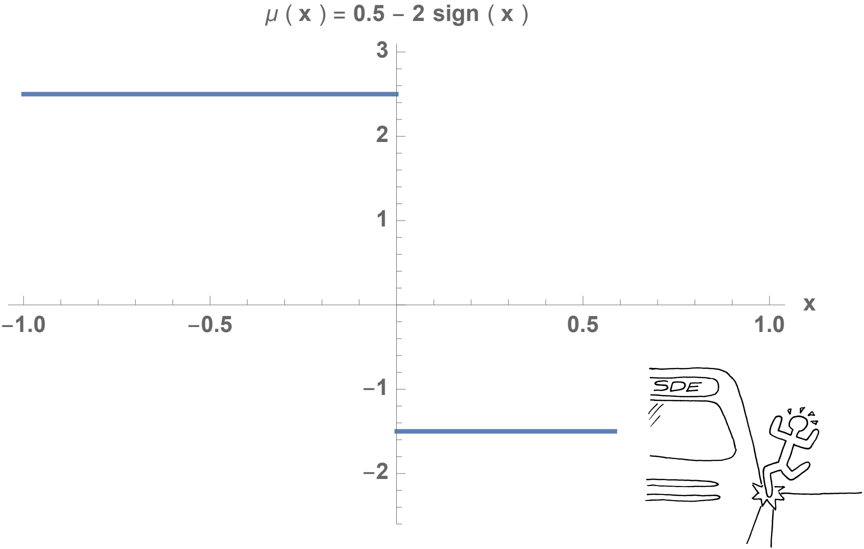
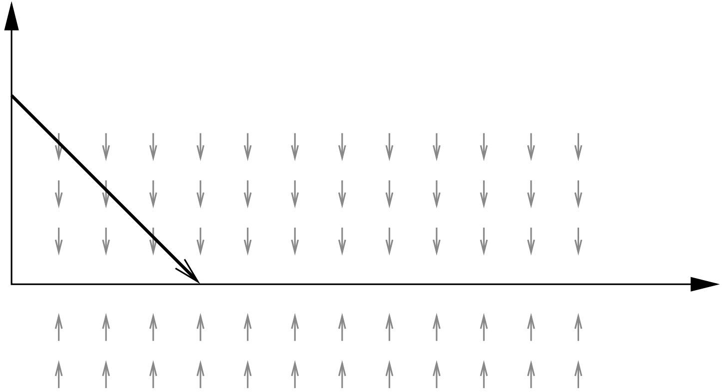


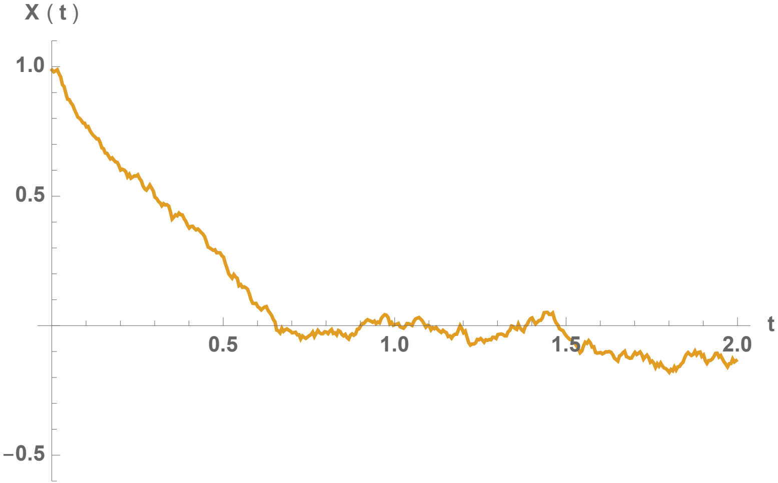
Can discontinuities matter?

$$dx_t = (0.5 - 2\ \mathrm{sign}(x_t))dt$$
$$x_0 = 1$$



Let us add noise again

$$dX_t = (0.5 - 2\,\mathrm{sign}(X_t))dt + dB_t$$
$$X_0 = 1$$



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There are SDEs with smooth and bounded but non-Lipschitz coefficients, for which there exists no numerical method that converges in finite time!
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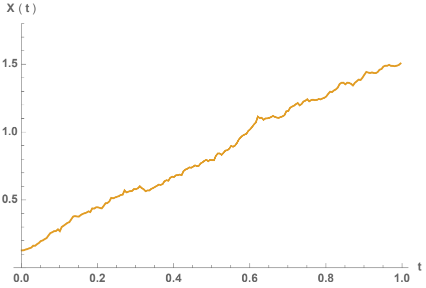

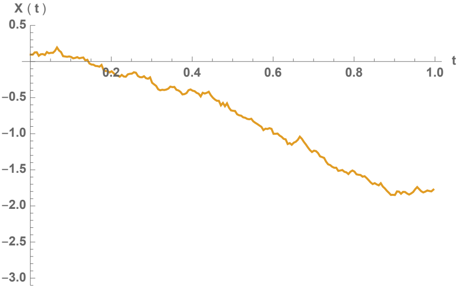

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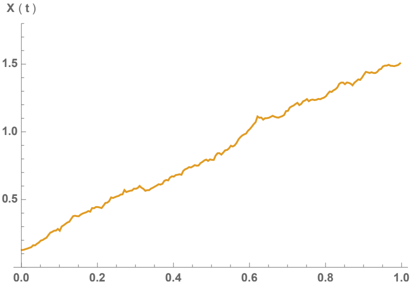
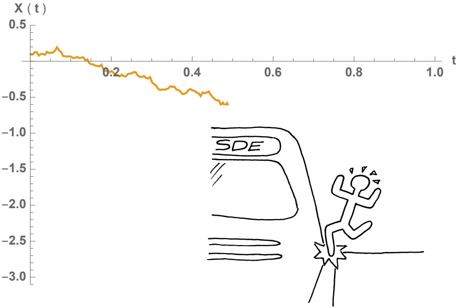
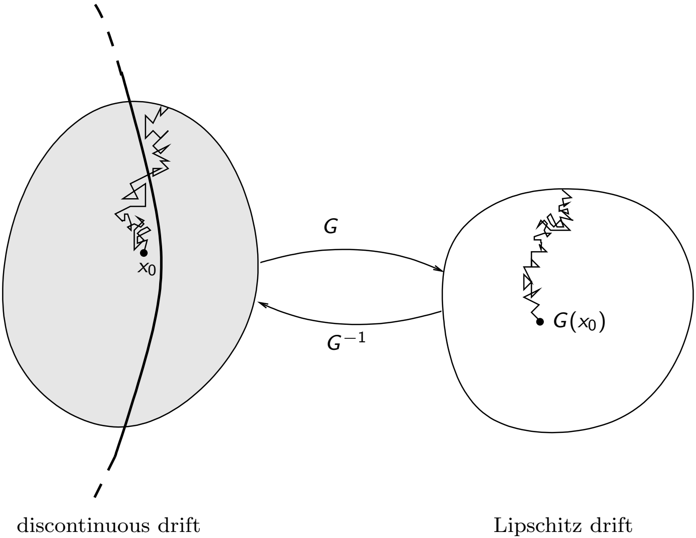




An idea



Assumptions

1. $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times d}$ is Lipschitz
2. $\mu : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is piecewise Lipschitz with exceptional set Θ
3. Θ is a C^4-hypersurface
4. *positive reach property:* there is $\varepsilon > 0$ s.t. for every $x \in \mathbb{R}^d$ with $d(x, \Theta) < \varepsilon$ there is a unique $p \in \Theta$ with $d(x, \Theta) = \|x - p\|$
5. *non-parallelity condition:* $\|\sigma(\xi)^\top n(\xi)\|^2 \geq c_0 > 0$ for all $\xi \in \Theta$
6. μ, σ bounded close to Θ
7. mild additional regularity of μ, σ close Θ and bounded n'', n'''

The transformation method

Algorithm (Leobacher, S. 2015, 2017)

- *Construct* $G = G_{\mu,\sigma}$ *and compute* G^{-1}
- *Define* $Z = G(X)$
- *Apply Euler-Maruyama to compute* $\hat{Z}$
- *Calculate* $\bar{X} = G^{-1}(\hat{Z})$

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The transformation method has strong convergence order $1/2$.

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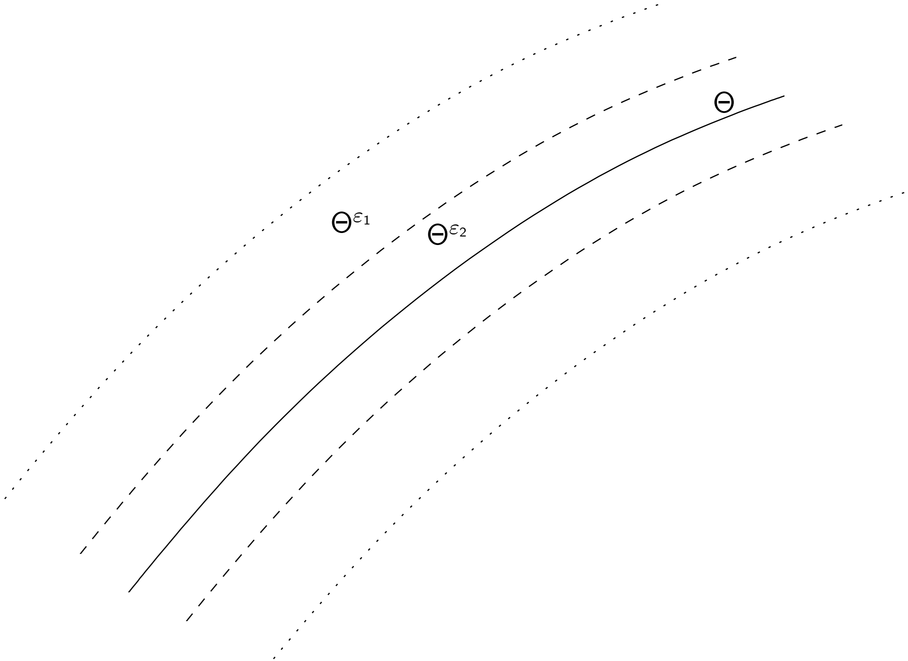
Remember $Z = G(X)$

$$\mathbb{E}\Big[\|X_t - \hat{X}_t\|^2\Big] = \mathbb{E}\Big[\|G^{-1}(Z_t) - G^{-1}(G(\hat{X}_t))\|^2\Big]$$
$$\leq 2(L_{G^{-1}})^2\mathbb{E}\Big[\|Z_t - \hat{Z}_t\|^2\Big] + 2(L_{G^{-1}})^2\mathbb{E}\Big[\|\hat{Z}_t - G(\hat{X}_t)\|^2\Big]$$

Theorem (Leobacher, S. 2018)

The Euler-Maruyama method has essentially strong convergence order $1/4$.

Can we do better?



An adaptive Euler-Maruyama method

For a stepsize function $h : \mathbb{R}^d \times [0,1] \longrightarrow [0,1]$:

$$\tau_0 = 0\,,$$
$$\tau_{k+1} = \tau_k + h(\hat{X}_{\tau_k}, \delta)\,,$$

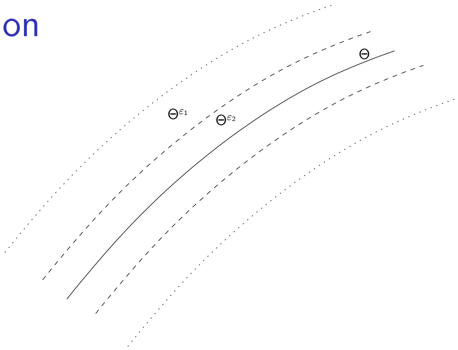
and

$$\hat{X}_0 = \xi\,,$$
$$\hat{X}_{\tau_{k+1}} = \hat{X}_{\tau_k} + \mu(\hat{X}_{\tau_k})(\tau_{k+1} - \tau_k) + \sigma(\hat{X}_{\tau_k})(B_{\tau_{k+1}} - B_{\tau_k})\,.$$

Computational cost of $\hat{X}$ proportional to

$$N(h) = \min\{k \in \mathbb{N} :\ \tau_k \geq T\}\,.$$

Stepsize selection



Choosing the stepsize in the middle regime optimally yields

$$h(x,\delta) = \begin{cases} \delta^2 & \text{if} \quad x \in \Theta^{\varepsilon_2} \\ \left(\frac{d(x,\Theta)}{\sup_{x\in\Theta^{\varepsilon_0}} \|\sigma(x)\|\cdot\log(\delta)}\right)^2 & \text{if} \quad x \in \Theta^{\varepsilon_1} \setminus \Theta^{\varepsilon_2} \\ \delta & \text{else} \end{cases}$$

with

$$\varepsilon_1 = \sup_{x\in\Theta^{\varepsilon_0}} \|\sigma(x)\| \cdot \log(1/\delta)\sqrt{\delta}, \qquad \varepsilon_2 = \sup_{x\in\Theta^{\varepsilon_0}} \|\sigma(x)\| \cdot \log(1/\delta)\delta\,.$$

So can we do better?

Theorem (Neuenkirch, S., Szpruch (2019))

For all $\epsilon > 0$ *there exist constants* $C_1, C_2 > 0$ *such that*

$$\mathbb{E}\left[\|X_T - \hat{X}_T\|^2\right]^{1/2} \leq C_1 \cdot (\max\{t_{k+1} - t_k\})^{1/2-\epsilon},$$

and

$$\mathbb{E}[N(h)] \leq C_2 \cdot (\max\{t_{k+1} - t_k\})^{-1+\epsilon}.$$

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Theorem (Müller-Gronbach, Yaroslavtseva 2019+)

In 1D the Euler-Maruyama method has strong convergence order $1/2 - \epsilon$.

Convergence rates via Sobolev regularity

Now we consider SDEs of the form

$$X_t = \xi + \int_0^t \mu(X_s)ds + W_t, \quad t \in [0, T],$$

where $\mu\colon \mathbb{R} \to \mathbb{R}$ is the (not necessarily continuous) drift coefficient.
For the case where μ is bounded and measureable, Zvonkin (1974) proves existence and uniqueness of a strong solution.

Assumptions

Assume that μ can be decomposed into a regular and an irregular part $a, b \colon \mathbb{R} \to \mathbb{R}$, that is $\mu = a + b$, satisfying the following assumptions:

1. (boundedness) a, b are bounded
2. (regular part) $a \in C_b^2(\mathbb{R})$, i.e. a is twice continuously differentiable with bounded derivatives
3. (irregular part-1) $b \in L^1(\mathbb{R})$

A novel framework for the error analysis

We show how to decompose the error into a discretization error and an error coming from approximating a quadrature problem for Brownian motion.

Theorem (Neuenkirch, S. (2019+))

For all $\varepsilon \in (0,1)$, *there exists* $c_d > 0$ *such that*

$$\mathbb{E}\left[|X_T - \hat{X}_T|^2\right] \leq c_d \cdot \left((\max\{t_{k+1} - t_k\})^2 + \mathcal{W}^{1-\varepsilon}\right),$$

where

$$\mathcal{W} = \mathbb{E}\left[\left|\int_0^T G'(W_s + \xi)\left(b(W_s + \xi) - b(W_{\underline{s}} + \xi)\right) ds\right|^2\right]$$

and where G *is a Zvonkin-type transform for the irregular part of the drift.*

A possible framework for b

4. (irregular part-2) there exists $\kappa \in (0,1)$ such that

$$|b|_\kappa := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|b(x) - b(y)|^2}{|x-y|^{2\kappa+1}} \, dx \, dy \right)^{1/2} < \infty,$$

i.e. b belongs to the fractional Sobolev-Slobodeckij space of order κ

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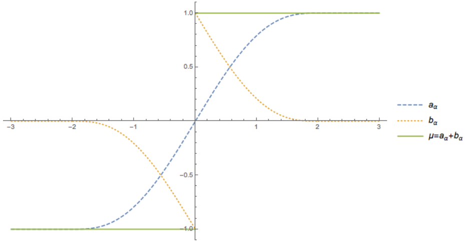
Example (Sign function)

Let $\mu(x) = \text{sign}(x)$. A decomposition $a, b \colon \mathbb{R} \to \mathbb{R}$, $\mu(x) = a(x) + b(x)$, which satisfies the assumptions for all $\kappa < 1/2$, can be chosen as



Analysis of the quadrature problem

Remember that the total error estimate is of the form

$$\mathbb{E}\left[\|X_T - \hat{X}_T\|^2\right] \leq c_d \cdot \left((\max\{t_{k+1} - t_k\})^2 + \mathcal{W}^{1-\varepsilon}\right).$$

Theorem (Neuenkirch, S. (2019+))

There exists a constant $c_q > 0$ such that

$$\mathcal{W} \leq c_q \cdot \frac{\log(n)}{n^{1+\kappa}}.$$

Convergence rate via Sobolev regularity

Corollary (Neuenkirch, S. (2019+))

For all $\epsilon \in (0,1)$ there exists a constant $c > 0$ such that the Euler-Maruyama scheme satisfies

$$\left(\mathbb{E}\left[\|X_t - \hat{X}_t\|^2\right]\right)^{1/2} \leq c \cdot (\max\{t_{k+1} - t_k\})^{(1+\kappa)/2-\epsilon}.$$

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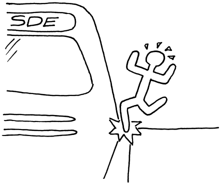
Example

For $\mu(x) = \text{sign}(x)$ the assumptions are satisfied for $\kappa < 1/2$ leading to

$$\left(\mathbb{E}\left[\|X_t - \hat{X}_t\|^2\right]\right)^{1/2} \leq c \cdot (\max\{t_{k+1} - t_k\})^{3/4-}.$$

Thank you for your attention!



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