Optimal Dividend and Investment Problems under Sparre Andersen Model

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Outline

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The Sparre Andersen Model (1957)

Assume that the reserve has the Cramér-Lundberg structure :

$$X_t = x + pt - Q_t := x + pt - \sum_{i=1}^{N_t} U_i, \quad t \in [0, T], \tag{1}$$

where

- $x = X_0 \geq 0$ — initial surplus,
- $p > 0$ — premium rate, and
- $Q_t := \sum_{i=1}^{N_t} U_i$ — claim process, in which the claim numbers (frequency) N is a *renewal* process (i.e., the interclaim times T_i's are i.i.d., but *not necessarily exponential*.)

The Sparre Andersen Model

Assume that the insurance company

- is allowed to invest but also pays dividends,
- the investment/dividend strategy $\pi \stackrel{\triangle}{=} (\gamma, L)$ is *self-financing*.

A "Toy" Model

$$\begin{cases} dX_t^\pi = [p + rX_t^\pi + (\mu - r)\gamma_t X_t]dt + \sigma\gamma_t X_t^\pi dB_t - dQ_t - dL_t, \\ X_0^\pi = x, \end{cases}$$

- $B = \{B_t\}_{t\geq 0}$ is a Brownian motion ;
- p, μ, r, σ are premium, appreciation, short rates, and volatility ;
- $\pi = (\gamma, L) \in \mathscr{U}_{ad} := \{\pi : \text{"adapted" and } \gamma_t \in [0,1],\ L_t \nearrow\}$.

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Main Objective

$$
\begin{aligned}
V(s,x) &:= \sup_{\pi\in\mathscr{U}_{ad}} J(s,x;\pi) \\
&= \sup_{\pi\in\mathscr{U}_{ad}} \mathbb{E}\Big\{\int_s^{\tau_s^\pi\wedge T} e^{-c(t-s)}dL_t \big| X_s^\pi = x\Big\},
\end{aligned}
\tag{2}
$$

- ◇ $\tau_s^\pi := \inf\{t \geq 0; X_t^\pi < 0\}$ is the *ruin time*,
- ◇ $T, c > 0$ are constants.

Main Features :

- "Singular-type" control problem (with jumps)
- "Endogenous" random terminal time (τ_s^π)
- Non-Markovian nature of the renewal process.

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What Do We Know ?

- **Cramér-Lundberg Model**
  - De Finetti (1957) — Expected cumul. dividend maximization
  - Gerber (1969) — existence, "band structure"
  - Asmussen-Taksar (1997) — diffusion approximation
    $\cdots$ -Taksar-XYZ- $\cdots$ (singular/impulse stochastic control)
  - Azcue-Muler (2005, 10)
    — reinsurance $\oplus$ dividend $\oplus$ investment (viscosity solution)
- **Sparre Andersen Model**
  - Li-Garrido ('04), Gerber-Shiu ('05) — ruin probability (Erlang)
  - Albrecher-Hartinger-Thonhauser ('06, '07) — Barrier "$+/-$"
- **Sparre Andersen $\oplus$ optimal dividend ($\oplus$ investment) ?**
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Main Difficulties

... ... In this context, a particular line of potential future research is to consider the optimal dividend problem when the Poisson claim number process is replaced by a general renewal process, i.e., the Sparre Andersen risk model.

... ... It is still an open problem to identify optimal dividend strategies in this model. One can morkovize the Sparre Andersen model by extending the dimension of the state space of the risk process, taking into account the time that has elapsed since the last claim occurrence. A reasonable strategy should also depend on this additional variable. But correspondingly also the dimension of the associated HJB equation will be extended which considerably increase the difficulties one is facing when analytically approaching this equation.

— H. Albrecher and S. Thonhauser,
Optimality Results for Dividend Problems in Insurance,
Rev. R. Acad. Cien. Seri A. Mat. Vol 103(2) 2009, 291-320.

Our Understanding

- Since the renewal process N (hence Q) is only *Semi-Markov*, the *Dynamic Programming* approach requires a serious look.
- As a semi-Markov process, Markovization is possible. I.e., by adding a "*random clock*" measuring the starting time *after* the last claim. Consequently, the so-called *delayed renewal process* should naturally come into play.
- It is tempting to somehow convert the problem to one with a deterministic terminal time, as was commonly done in stochastic control problem with random horizon ...
- ... but in this case the terminal (ruin) time depends on the strategy !

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A Fact for Renewal/Semi-Markov Process

Backward Markovization

If we define

$$W_t = t - \sigma_{N_t}, \qquad t \geq 0, \tag{3}$$

where σ_{N_t} is the last jump time before t. Then

- (t, N_t, W_t), $t \geq 0$, is a *Piecewise Deterministic Markov Process* (PDMP), and
- $0 \leq W_t \leq t \leq T$, for $t \in [0, T]$.

For $\pi \in \mathscr{U}_{ad}$, we consider the controlled dynamics for $t \in [s, T]$:

$$\begin{cases} dX_t^\pi = pdt + rX_t^\pi dt + \sigma\gamma_t X_t^\pi dB_t - dQ_t - dL_t, & X_s^\pi = x; \\ dW_t := d(t - \sigma_{N_t}), & W_s = w. \end{cases} \tag{4}$$

The Canonical Set-up

Consider the *canonical* (filtered) probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$:

- $\Omega = \mathbb{D}^3_T = \mathbb{D}([0,T]; \mathbb{R}^3)$ — càdlàg functions on $[0, T]$;
- $\mathcal{F} = \mathscr{B}^3_T = \mathscr{B}(\mathbb{D}^3_T)$, $\mathbb{F} = \{\mathscr{B}^3_t\}_{t\in[0,T]}$;
- Let $(B, N, W)_t(\omega) := (\omega^1(t), \omega^2(t), \omega^3(t))$, $t \in [0, T]$, $\omega = (\omega^1, \omega^2, \omega^3) \in \Omega$ be the canonical process ;
- Consider $\mathbb{P} \in \mathscr{P}(\Omega)$ such that under $\mathbb{P}$,
  - B is a Brownian motion ;
  - N is a *renewal* process , $\perp\!\!\!\perp B$, with jump times $\{\sigma_n\}_{n=1}^{\infty}$;
  - $W_t = t - \sigma_{N_t}$, $t \in [0, T]$
  - U_i's are i.i.d., $\perp\!\!\!\perp (B, N, W)$.

Furthermore...

- Assume that $\{T_i\} = \{\sigma_i - \sigma_{i-1}\}$, the i.i.d. "interclaims" of N, have the common distribution F and density f. Then, with $\lambda(t) := f(t)/\bar{F}(t)$, $t \geq 0$, one has

$$\mathbb{P}\{T_1 > t\} := \bar{F}(t) = 1 - F(t) = e^{-\int_0^t \lambda(u)du}.$$

- Let $\mathbb{P}_{sw}(\cdot) := \mathbb{P}[\,\cdot\,|W_s = w]$ be the *regular conditional probability distribution* (RCPD) on $(\Omega, \mathcal{F})$. Then, under $\mathbb{P}_{sw}$,
  - $B_t^s := B_t - B_s$, $t \geq s$ is a BM on $[s, T]$;
  - $N_t^{s,w} := N_t - N_s$, $t \geq s$ is a *delayed* renewal process with $\mathbb{P}_{sw}\{T_1^{s,w} > t\} = \exp\{-\int_w^{w+t} \lambda(u)du\}$;
  - $W_t^{s,w} := w + (t - s) + (\sigma_{N_t} - \sigma_{N_s})$, $t \geq s$, $\mathbb{P}_{sw}$-a.s.

More precisely ...

- $J(s,x,w;\pi) := \mathbb{E}_{sxw}\Big\{ \int_s^{\tau_s^\pi \wedge T} e^{-c(t-s)} dL_t \Big\} = \mathbb{E}_{sw}\{\cdot | X_s^\pi = x\}$,
- $V(s,x,w) := \sup_{\pi \in \mathscr{U}_{ad}[s,T]} J(s,x,w;\pi)$.

Main Assumptions

- $V(s,x,w) = 0$, for $(s,x,w) \notin D$, where

$$D \stackrel{\triangle}{=} Dom(V) = \{(s,x,w) : 0 \le s \le T, x \ge 0, 0 \le w \le s\}.$$

- *The dividend process L is of the form $L_t = \int_0^t a_s ds$, $t \ge 0$, where $a \in L^0_{\mathbb{F}}([0,T])$, s.t. $0 \le a_t \le M$, for some $M \ge p > 0$.*

Note : The boundedness of a_t excludes all "singular" type of L! (See, e.g., Asmussen-Taksar (97), Schäl (98), Gerber-Shiu (06), ...).

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$\mathscr{U}_{ad}$ Fine Tuned

- Note that if $Q_t^{s,w} \triangleq \sum_{i=1}^{N_t^{s,w}} U_i \equiv 0$, then for given $\pi = (\gamma, a) \in \mathscr{U}_{ad}$, the solution to the linear SDE(4) is

$$X_t^\pi = Z_t^s \Big[X_s^\pi + \int_s^t [Z_u^s]^{-1}(p - a_u)du \Big], \quad t \in [s, T], \qquad (5)$$

where $Z_t^s := \exp\Big\{ r(t-s) + \sigma \int_s^t \gamma_u dB_u - \frac{\sigma^2}{2} \int_s^t |\gamma_u|^2 du \Big\}$.

- We shall require that $(p - a_t)\mathbf{1}_{\{X_t^\pi = 0\}} \geq 0$, so that without the claims (i.e., $Q^{s,w} \equiv 0$), one has $dX_t^\pi \geq 0$ on $\{X_t^\pi = 0\}$ $\Longrightarrow X_t^\pi \geq 0, \quad t \geq 0$, $\mathbb{P}$-a.s. (In other words, the bankruptcy should **not** be caused by paying too much dividend at $X_t = 0$.)

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- In fact, we can show that if $\pi \in \mathscr{U}_{ad}$ is such that $\mathbb{P}\{\sigma_i \wedge T < \tau^\pi < \sigma_{i+1} \wedge T\} > 0$, for some i, then there exists $\tilde{\pi} \in \mathscr{U}_{ad}$ s.t. $\mathbb{P}\{\tau^{\tilde{\pi}} \in \bigcup_{i=1}^{\infty} \sigma_i\} = 1$, and

$$J(s, x, w; \tilde{\pi}) > J(s, x, w; \pi).$$

— *Bankrupting oneself by paying dividend is never optimal!*

- we can/shall fine-tune the *admissible control set* as :

$$\tilde{\mathscr{U}}_{ad} := \Big\{\pi = (\gamma, a) \in \mathscr{U}_{ad} : \Delta X^{\pi}_{\tau^\pi} \mathbf{1}_{\{\tau^\pi < T\}} < 0, \mathbb{P}\text{-a.s.}\Big\}. \quad (6)$$

The set $\tilde{\mathscr{U}}_{ad}[s, T]$ is defined similarly, and we often denote $\tilde{\mathscr{U}}_{ad} = \mathscr{U}_{ad}$.

Outline

Basic Properties of $V(s,x,w)$

- $V(s,x,w)$ is increasing with respect to x;
- $V(s,x,w) \leq \frac{M}{c}(1-e^{-c(T-s)})$ for any $(s,x,w) \in D$, where $M > 0$ is the given bound of the dividend rate; and
- $\lim_{x\to\infty} V(s,x,w) = \frac{M}{c}[1-e^{-c(T-s)}]$, for $0 \leq s \leq T$, $0 \leq w \leq s$.

Remark : (i) and (ii) are straightforward. (iii) comes from (ii) and a simple calculation of $\lim_{x\to\infty} J(s,x,w,\pi^0)$, where $\pi^0 \equiv (0,M)$.

Continuity of $s \mapsto V(s,x,w)$

- **Main Result**
  - $V(s+h,x,w) - V(s,x,w) \le 0$;
  - $V(s,x,w) - V(s+h,x,w) \le Mh$.

  Here where $M > 0$ is the given bound of the dividend rate, $h > 0$, s,t, (s,x,w), $(s+h,x,w) \in D$.

- **Main Difficulty :**
  - How to "freeze" w while "moving" s? (W is a "clock"!)

- **Main Idea :** —*Time Shifting*
  - E.g., $s = w = 0$. $\forall \pi \in \mathscr{U}_{ad}^{h,0}[h,T]$, define $\bar{\pi}_t^h = \pi_{h+t} = \eta(\cdots)$, and then define $\tilde{\pi}_t^h := \eta(t,(B,Q,W)_{\cdot\wedge t\wedge(T-h)}) \in \mathscr{U}_{ad}[0,T]$.
  - Using uniqueness (in law) to show that

$$J_{h,T}(h,x,0;\pi) = \bar{J}_{0,T-h}(0,x,0;\bar{\pi}^h) = J_{0,T}(\cdots\tilde{\pi}^h) \le V(0,x,0).$$

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Continuity of $x \mapsto V(s,x,w)$

- **Main Result :**
  - For any compact set $K \subset D$, the mapping $x \mapsto V(s,x,w)$ is continuous, uniformly for $(s,x,w) \in K$.
- **Main Difficulty :**
  - The mapping $x \mapsto \tau^{\pi,x}$ is **discontinuous** in general !
- **Main Idea :** — *Penalization Method*
  - Define $\beta^{\pi,x}(t,\varepsilon) := e^{-\frac{1}{\varepsilon}\int_s^t (X_r^{\pi,x})^- dr}$, $\pi \in \mathscr{U}_{ad}[s,T]$, $t \geq s$, and

$$\begin{cases} J^\varepsilon(s,x,w;\pi) := \mathbb{E}_{sw}\Big[\displaystyle\int_s^T \beta^{\pi,x}(t,\varepsilon)e^{-c(t-s)}a_t dt\Big], \\ V^\varepsilon(s,x,w) = \sup_{\pi\in\mathscr{U}_{ad}} J^\varepsilon(s,x,w;\pi). \end{cases}$$

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Continuity of $x \mapsto V(s,x,w)$.

- Then $V^\varepsilon(s,\cdot,w)$ is continuous, uniformly on any $K \subset\subset D$, and

$$V^\varepsilon(s,x,w) = \sup_\pi \mathbb{E}\Big[\int_s^{\tau_s^\pi} e^{-c(t-s)} a_t dt + \int_{\tau_s^\pi}^T (\cdots)\Big] \geq V(s,x,w).$$

- Let $D_\theta := \{(s,x,w) \in [0,T] \times (-\theta,\infty) \times [0,s]\}$, $\theta > 0$; and $h^\theta(\varepsilon) := V^\varepsilon(\tau^\theta, X^\pi_{\tau^\theta}, W_{\tau^\theta})$, where τ^θ is the exit time from D_θ.
- Since $D_\theta \searrow D$, $\tau^\theta \geq \tau$. Applying DPP to V^ε to obtain

$$\begin{aligned} V^\varepsilon(s,x,w) &= \sup_\pi \mathbb{E}_{sxw}\Big\{\int_s^\tau e^{-c(t-s)} a_t dt + \int_\tau^{\tau^\theta} \beta(t,\varepsilon)(\cdots)dt \\ &\qquad + e^{-(\tau^\theta - s)}\beta(\tau^\theta,\varepsilon) V^\varepsilon(\tau^\theta, X^\pi_{\tau^\theta}, W_{\tau^\theta})\Big\} \\ &\leq V(s,x,w) + C \sup_\pi \mathbb{E}_{sxw}|\tau^\theta - \tau| + h^\theta(\varepsilon). \qquad \blacksquare \end{aligned}$$

Continuity of $w \mapsto V(s, x, w)$

- We show two inequalities : $\forall h > 0$, s.t., $0 \le s < s + h < T$,

$$\begin{cases} V(s+h, x, w+h) - V(s, x, w) \\ \qquad \le \left[1 - e^{-(ch + \int_w^{w+h} \lambda(u)du)}\right] V(s+h, x, w+h); \\ V(s, x, w+h) - V(s, x, w) \\ \qquad \le Mh + \left[1 - e^{-(ch + \int_w^{w+h} \lambda(u)du)}\right] V(s+h, x, w+h). \end{cases}$$

$\Longrightarrow \lim_{h \downarrow 0}[V(s+h, x, w+h) - V(s, x, w)] = 0$, uniformly in $(s, x, w) \in D$.

- This, together with continuity in s, shows that V is uniformly continuous in w, uniformly on K. ■

Dynamic Programming Principle (DPP)

- **Want :** $\forall (s,x,w) \in D$ and any *stopping time* $\tau \in [s,T]$,

$$V(s,x,w) = \sup_{\pi \in \mathscr{U}_{ad}[s,T]} \mathbb{E}_{sxw}\Big\{ \int_s^{\tau\wedge\tau^\pi} e^{-c(t-s)} a_t dt \quad (7)$$
$$+ e^{-c(\tau\wedge\tau^\pi - s)} V(R^\pi_{\tau\wedge\tau^\pi}) \Big\},$$

where $R^\pi_t := (t, X^\pi, W^\pi)$, and τ^π is the ruin time of X^π.

- **Main Difficulty :** Continuity of $(s,x,w) \mapsto J(s,x,w;\pi)$?
- **Main Result :**
  - $\forall \varepsilon > 0, \exists \delta > 0$ (ind. of $(s,x,w) \in D$), s. t. $\forall \pi \in \mathscr{U}_{ad}[s,T]$ and $h := (h_1, h_2)$ with $0 < h_1, h_2 < \delta$, $\exists \hat{\pi}^h \in \mathscr{U}_{ad}[s,T]$,

$$J(s,x,w,\pi) - J(s, x-h_1, w-h_2, \hat{\pi}^h) \le \varepsilon, \quad \forall (s,x,w) \in D. \quad (8)$$

  - The usual proof of "partitioning space" can now go through. ∎

Dynamic Programming Principle (DPP)

- **Want :** $\forall(s,x,w) \in D$ and any *stopping time* $\tau \in [s, T]$,

$$V(s,x,w) = \sup_{\pi \in \mathscr{U}_{ad}[s,T]} \mathbb{E}_{sxw}\Big\{ \int_s^{\tau\wedge\tau^\pi} e^{-c(t-s)} a_t dt + e^{-c(\tau\wedge\tau^\pi - s)} V(R^\pi_{\tau\wedge\tau^\pi}) \Big\}, \quad (7)$$

  where $R^\pi_t := (t, X^\pi, W^\pi)$, and τ^π is the ruin time of X^π.

- **Main Difficulty :** Continuity of $(s,x,w) \mapsto J(s,x,w;\pi)$?

- **Main Result :**
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Outline

The Hamiltonian

- Define

$$H(s,x,w,u,\xi,A,z,\pi) \quad := \quad \frac{\sigma^2}{2}\gamma^2 x^2 A + (p + rx - a)\xi^1 + \xi^2$$
$$+\lambda(w)z + (a - cu),$$

where $\xi = (\xi^1, \xi^2)$ and $\pi = (\gamma, a) \in [0,1] \times [0,M]$.

- For $\varphi \in \mathbb{C}^{1,2,1}(D)$, we define the following Hamiltonian :

$$\mathscr{H}(s,x,w,\varphi,\nabla\varphi,\varphi_{xx},\pi) := H(s,x,w,\varphi,\nabla\varphi,\varphi_{xx},I(\varphi),\pi),$$

  - $\nabla\varphi := (\varphi_x, \varphi_w)$; $I[\varphi] := \int_0^\infty [\varphi(s,x-u,0) - \varphi(s,x,w)]dG(u)$.
  - Note the intrinsic *degeneracy* of $\mathscr{H}$, as a second order differential operator (no φ_{ww}) !

HJB Equation

Consider the following HJB equation :

$$\begin{cases} \{V_s + \mathscr{L}[V]\}(s,x,w) = 0; & (s,x,w) \in \mathrm{int}D; \\ V(s,x,w) = 0, & (s,x,w) \in D^c \cup \{s = T\}, \end{cases} \tag{9}$$

where for $\varphi \in \mathbb{C}^{1,2,1}(D)$,

$$\mathscr{L}[\varphi](s,x,w) := \sup_{\pi \in [0,1]\times[0,M]} \mathscr{H}(s,x,w,\varphi,\nabla\varphi,\varphi_{xx},\pi). \tag{10}$$

Note :

- Since $V \equiv 0$ on D^c, it seems very unlikely to have a global "classical" solution to the HJB equation (9) !
- Will denote $\mathscr{D} := \mathrm{int}D$, $\mathscr{D}^* := D \setminus \{s = T\}$, $\partial\mathscr{D}^* := \mathscr{D}^* \setminus \mathscr{D}$, and $\mathbb{C}_0^{1,2,1}(D) := \{\varphi \in \mathbb{C}^{1,2,1}(D) : \varphi \equiv 0 \text{ on } D^c\}$.

Boundary Behavior of V (on $\partial\mathscr{D}^*$)

- Let $V, \varphi \in \mathbb{C}_0^{1,2,1}(D)$, such that V satisfies (9), and

$$0 = [V - \varphi](s, 0, w) = \max_{(t,y,v)\in\mathscr{D}^*} [V - \varphi](t, y, v), \ (s, 0, w) \in \partial\mathscr{D}^*.$$

- Then, denoting $\nabla = (\partial_x, \partial_w)$,
  - $(\partial_t, \nabla)(V - \varphi)(s, 0, w) = \alpha\nu$, $\alpha > 0$, where $\nu = (0, -1, 0)$ is the *outward normal* of $D = \overline{\mathscr{D}}^*$ at $\{x = 0\}$, and
  - $I[V - \varphi](s, 0, w) = -[V - \varphi](s, 0, w) = 0$.

- Thus, for any $\pi = (\gamma, a) \in [0, 1] \times [0, M]$ we have

$$\begin{aligned} & [\varphi_s + \mathscr{H}(\cdot, \varphi, \nabla\varphi, \varphi_{xx}, I(\varphi), \pi)](s, 0, w) \qquad (11) \\ = \ & [V_s + \mathscr{H}(\cdot, V, \nabla V, V_{xx}, I(V), \pi)](s, 0, w) + \alpha(p - a). \end{aligned}$$

Boundary Behavior of V (on $\partial\mathscr{D}^*$)

- Thus, assume $a \le p$ (naturally, since $x = 0\,!$) we have

$$\{\varphi_s + \mathscr{L}[\varphi]\}(s,0,w) \ge \{V_s + \mathscr{L}[V]\}(s,0,w) = 0, \qquad (12)$$

- For the other boundaries $\{w = 0\}$ and $\{w = s\}$, we note that
  - $[V_{xx} - \varphi_{xx}] \le 0$, and
  - the outward normals are $\nu = (0,0,-1)$ and $(-1,0,1)$, resp.
- A similar calculation as (11) would lead to (12) in both cases.

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Constrained Viscosity Solution

Definition

Let $\mathcal{O} \subseteq \mathscr{D}^*$ be s.t. $\partial_T \mathcal{O} := \{(T, y, v) \in \partial \mathcal{O}\} \neq \emptyset$, and $v \in \mathbb{C}(\mathcal{O})$.

- v is called a viscosity sub-(reps. super-)solution of (9) on $\mathcal{O}$, if
  - $v(T, y, v) \leq$ (resp. $\geq$) 0, for $(T, y, v) \in \partial_T \mathcal{O}$;
  - for any $(s, x, w) \in \mathcal{O}$ and $\varphi \in \mathbb{C}_0^{1,2,1}(\mathcal{O})$, such that

  $$0 = [v - \varphi](s, x, w) = \max_{(t,y,v)\in\mathcal{O}} (\text{resp.} \min_{(t,y,v)\in\mathcal{O}})[v - \varphi](t, y, v),$$

  it holds that

  $$\{\varphi_s + \mathscr{L}[\varphi]\}(s, x, w) \geq (\text{resp.} \leq)0. \tag{13}$$

- $v \in \mathbb{C}(D)$ is called a *constrained viscosity solution* of (9) on $\mathscr{D}^*$ if it is both a subsolution on $\mathscr{D}^*$ and a supersolution on $\mathscr{D}$.

An Equivalent Definition

Definition

Let $\mathcal{O} \subseteq \mathscr{D}^*$, $u \in \mathbb{C}(\mathcal{O})$, and $(s, x, w) \in \mathcal{O}$.

- The set of *parabolic super-jets* of u at (s, x, w), denoted by $\mathscr{P}_{\mathcal{O}}^{+(1,2,1)} u(s, x, w)$, is defined as all $(q, \xi, A) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ s.t. for $(s, X) := (s, x, w), (t, Y) := (t, y, v) \in \mathcal{O}$, it holds

$$u(t, Y) \leq u(s, X) + q(t - s) + (\xi, X - Y) + \frac{1}{2} A(x - y)^2 + o(|t - s| + |w - v| + |y - x|^2), \qquad (14)$$

- The set of *parabolic sub-jets* of u at $(s, x, w) \in \mathcal{O}$, denoted by $\mathscr{P}_{\mathcal{O}}^{-(1,2,1)} u(s, x, w)$, is the set of all $(q, \xi, A) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ such that (14) holds with "$\leq$" being replaced by "$\geq$".

An Equivalent Definition

Definition

Let $\mathscr{O} \subseteq \mathscr{D}^*$.

- $\underline{u} \in \mathbb{C}(\mathscr{O})$ (resp. $\bar{u} \in \mathbb{C}(\mathscr{O})$) is a viscosity sub-(resp. super-) solution of (9) on $\mathscr{O}$, if for any $(s, x, w) \in \mathscr{O}$, it holds that

$$\begin{array}{rrcl} & q + \sup_{\pi \in [0,1]\times[0,M]} H(s, x, w, \underline{u}, \xi, A, I[\underline{u}], \pi) & \geq & 0 \\ \text{(resp.} & q + \sup_{\pi \in [0,1]\times[0,M]} H(s, x, w, \bar{u}, \xi, A, I[\bar{u}], \pi) & \leq & 0), \end{array}$$

$\forall (q, \xi, A) \in \mathscr{P}_{\mathscr{O}}^{+(1,2,1)} \underline{u}(s, x, w)$ (resp. $\mathscr{P}_{\mathscr{O}}^{-(1,2,1)} \bar{u}(s, x, w)$).

- We say that u is a "*constrained viscosity solution*" of (9) on $\mathscr{D}^*$ if it is both a subsolution on $\mathscr{D}^*$, and a supersolution on $\mathscr{D}$.

Supersolution

Given $(s,x,w) \in \mathscr{D}$. Let $\varphi \in \mathbb{C}_0^{1,2,1}(D)$ be such that $V - \varphi$ attains its minimum at (s,x,w) with $\varphi(s,x,w) = V(s,x,w)$

- Let $s < s+h < T$, and denote $\tau_s^h := s + h \wedge T_1^{s,w}$. Then there is no jump on $[0, \tau_s^h)$.
- By DPP (Theorem 4), for any $\pi = (\gamma, a) \in \mathscr{U}_{ad}[s,T]$,

$$
\begin{aligned}
0 \;&\geq\; \mathbb{E}_{sxw}\Big[\int_s^{\tau_s^h} e^{-c(t-s)} a_t dt + e^{-c(\tau_s^h - s)} V(R^\pi_{\tau_s^h})\Big] - V(s,x,w) \\
&\geq\; \mathbb{E}_{sxw}\Big[\int_s^{\tau_s^h} e^{-c(t-s)} a_t dt + e^{-c(\tau_s^h - s)} \varphi(R^\pi_{\tau_s^h})\Big] - \varphi(s,x,w).
\end{aligned}
$$

where $R_t^\pi := (t, X_t^{\pi,s,w,x}, W_t^{s,w})$.

Supersolution

- Given $(\gamma, a) \in [0,1] \times [0,M]$, define a "feedback" strategy :

$$\pi_t^0 = (\gamma_0, a_t^0) := (\gamma, a\mathbf{1}_{\{t<\tau_0\}} + p\mathbf{1}_{\{t\geq\tau_0\}}), \quad t \geq s,$$

where $\tau_0 = \inf\{t > s, X_t^{\pi^0} = 0\}$. Then $\pi^0 \in \mathscr{U}_{ad}[s,T]$.

- Clearly, $R_t^{\pi^0} = (t, X_t^{\pi^0,s,w,x}, W_t^{s,w}) \in \mathscr{D}$, for $t \in [s, \tau_s^h)$, and
... analyzing $\mathbb{E}_{sxw}[\int_s^{\tau_s^h} e^{-c(t-s)} a_t^0 dt] \oplus$ applying Itô to $\varphi(R^{\pi^0})$
... ... we obtain

$$0 \geq \{\varphi_t + \mathscr{H}(\cdot, \varphi, \nabla\varphi, \varphi_{xx}, \gamma, a)\}(s,x,w). \tag{15}$$

- Since (γ, a) is arbitrary, V is a viscosity supersolution on $\mathscr{D}$.

Subsolution

Task : Show that V is a viscosity subsolution on $\mathscr{D}^*$.

Main Idea

Suppose not. Then (one can show)

- $\exists (s,x,w) \in \mathscr{D}^*$, $\psi \in \mathbb{C}_0^{1,2,1}(D)$, $\varepsilon > 0$, $\rho > 0$, s.t.
  $0 = [V - \psi](s,x,w) = \max_{(t,y,v)\in\mathscr{D}^*}[V-\psi](t,y,v)$,
- but it holds that

$$\begin{aligned} \{\psi_s + \mathscr{L}[\psi]\}(t,y,v) \le -\varepsilon c, \; (t,y,v) \in \overline{B_\rho \cap \mathscr{D}^*} \setminus \{t = T\}; \\ V(t,y,v) \le \psi(t,y,v) - \varepsilon, \quad (t,y,v) \in \partial B_\rho \cap \mathscr{D}^*, \end{aligned}$$

where B_ρ is an open ball centered at (s,x,w) with radius ρ.

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where B_ρ is an open ball centered at (s,x,w) with radius ρ.

Subsolution

- $\forall \pi = (\gamma, a) \in \mathscr{U}_{ad}$, let $\tau_\rho := \inf\{t > s : R_t \notin \overline{B_\rho \cap \mathscr{D}^*}\}$, where $R_t = (t, X_t^\pi, W_t^{s,w})$, and $\tau := \tau_\rho \wedge (s + T_1^{s,w})$.
- Since $V(R_t) \le \psi(R_t)$ for $t \le \tau$, applying Itô to $e^{-c\cdot}\psi$ one has

$$
\begin{aligned}
& \mathbb{E}_{sxw}\Big[\int_s^\tau e^{-c(t-s)} a_t dt + e^{-c(\tau-s)} V(R_\tau)\Big] \\
\le\ & \mathbb{E}_{sxw}\Big[\psi(s,x,w) - \varepsilon e^{-c(\tau_\rho - s)} \mathbf{1}_{\{\tau_\rho < s + T_1^{s,w}\}} \\
& + \int_s^\tau e^{-c(t-s)} [\psi_t + \mathscr{H}(\cdots, \gamma_t, a_t)(R_t)] dt\Big] \\
\le\ & \psi(s,x,w) - \varepsilon \mathbb{E}_{sxw}\left[e^{-c(\tau-s)} \mathbf{1}_{\{\tau_\rho < s + T_1^{s,w}\}} + (1 - e^{-c(\tau-s)})\right] \\
\le\ & V(s,x,w) - \varepsilon \mathbb{E}_{sxw}(1 - e^{-cT_1^{s,w}}) < V(s,x,w).
\end{aligned}
$$

$\Longrightarrow$ contradicts DPP.

Comparison Principle

Recall ...

The value function $V \geq 0$ satisfies the following Condition-(C) :

- V is uniformly continuous on D;
- the mapping $x \mapsto V(s,x,w)$ is increasing, and $\lim_{x\to\infty} V(s,x,w) = \frac{M}{c}[1 - e^{-c(T-s)}]$;
- $V(T,y,v) = 0$ for any $(y,v) \in [0,\infty) \times [0,T]$.

Theorem 5 (Comparison Principle)

Let $\underline{u}$ be a viscosity subsol. on $\mathscr{D}^*$ and $\bar{u}$ a viscosity supersol. on $\mathscr{D}$. If both $\bar{u}$ and $\underline{u}$ satisfy the Condition-(C), then $\underline{u} \leq \bar{u}$ on D.

Consequently, the value function is the unique constrained viscosity solution of (9) satisfying Condition (C) on D.

Main Idea of the Proof (of Theorem 5)

- First "massage" the supersolution $\bar{u}$ to

$$\bar{u}^{\rho,\theta,\varsigma}(t,y,v) = \rho\bar{u}(t,y,v) + \theta\frac{T-t+\varsigma}{t}, \qquad \rho > 1, \theta, \varsigma > 0.$$

- It suffices to show that
  - ◇ $\underline{u}(t,y,v) \le \bar{u}^{\rho,\theta,\varsigma}(t,y,v)$
  - ◇ on $\mathscr{D}_b := \{(t,y,v) : 0 < t < T, 0 \le y < b, 0 \le v \le t\}$.

- **Suppose not.** Then, noting that $\underline{u} - \bar{u}^{\rho,\theta,\varsigma} \le 0$ on $\{t = 0, T; y = b\}$, one can show that $\exists(t^*, y^*, v^*) \in \mathscr{D}_b^1$, s.t.

$$M_b := [\underline{u} - \bar{u}^{\rho,\theta,\varsigma}](t^*, y^*, v^*) > 0, \tag{16}$$

where $\mathscr{D}_b^1 := \bar{\mathscr{D}}_b \setminus \{t = 0, T; y = b\}$.

Then ...

- **Case 1.** [$(t^*, y^*, v^*) \in \mathscr{D}_b^0 := \mathrm{int}\mathscr{D}_b$.] For $\varepsilon > 0$, define

$$\Sigma_\varepsilon^b(t,x,w,y,v) = \underline{u}(t,x,w) - \bar{u}^{\rho,\theta,\varsigma}(t,y,v) - \frac{(x-y)^2+(w-v)^2}{2\varepsilon}.$$

  - Since $\mathscr{C}_b$ is compact, one can show that $\exists \varepsilon_0 > 0$,

$$(t_\varepsilon, x_\varepsilon, w_\varepsilon, y_\varepsilon, v_\varepsilon)) \in \mathrm{argmax}_{\mathscr{C}_b} \Sigma_\varepsilon^b \cap \mathrm{int}\mathscr{C}_b, \quad 0 < \varepsilon < \varepsilon_0.$$

  - Following the standard arguments (e.g., "User's guide") to shows that $(t_\varepsilon, x_\varepsilon, w_\varepsilon, y_\varepsilon, v_\varepsilon) \to (\bar{t}, (\bar{x}, \bar{w}), (\bar{x}, \bar{w})) \in \overline{\mathscr{C}}_b$, and use the subsolution property to derive the contradiction.

- **Case 2 :** [$(t^*, y^*, v^*) \in \partial\mathscr{D}_b^1$. (Hard !)]
  - **Main Idea :** Move the point (t^*, y^*, v^*) away from (possibly) the boundary of $\mathscr{D}_b^1$ into $\mathscr{D}_b^0$ and then argue as Case 1.
  - **Main Reference :** Soner ('86), Ishii-Lions ('90).

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Outline

Optimal Strategy

An "Educated Guess"

From the HJB equation one can find the "maximizer", which leads to the following feedback optimal control :

$$\begin{cases} \gamma_t = -\frac{(\mu - r)V_x(t, X_t^*, W_t)}{\sigma^2 x V_{xx}(t, X_t^*, W_t)} \wedge 1; \\ a_t = M\mathbf{1}_{\{V_x(t, X_t^*, W_t) < 1\}}. \end{cases}$$

Issues need to be addressed :

- Regularity of the value function ?
- Wellposedness of the closed-loop system ?
- ... ...
- Existence of the optimal strategy ?

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Since the "classical solution" for the HJB equation is essentially impossible for two reasons :

- Boundary condition ($V = 0$ on D^c)
- Degeneracy of $\mathscr{L}$ (there is no V_{ww} !)

a more practical goal would be to find the reasonable approximation of V which could lead to the "*ε-optimal strategy*".

Main Idea

- Find an approximating stochastic control problem whose value function V^ε is more regular
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A Candidate Approximating Problem

$$\begin{cases} dX_t^\pi = pdt + rX_t^\pi dt + \sigma\gamma_t X_t^\pi dB_t - dQ_t - dL_t, & X_s^\pi = x; \\ dW_t^\varepsilon := \sqrt{\varepsilon} dB_t^1 + d(t - \sigma_{N_t}), & W_s^\varepsilon = w. \end{cases} \tag{17}$$

where $B^\varepsilon \perp\!\!\!\perp B$, and $\varepsilon > 0$. Note that the domain of this problem should be $\tilde{D} := [0, T] \times [0, \infty) \times \mathbb{R}$, so that "exist of $\tilde{D}$" = "ruin" !

Main Difficulties

- The analysis of "perturbed delayed renewal process" W^ε (e.g., jump intensity ?)
- Regularity of V^ε ?
- $V^\varepsilon \to V$? In what sense ?

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where $B^\varepsilon \perp\!\!\!\perp B$, and $\varepsilon > 0$. Note that the domain of this problem should be $\tilde{D} := [0, T] \times [0, \infty) \times \mathbb{R}$, so that "exist of $\tilde{D}$" = "ruin" !

Main Difficulties

- The analysis of "perturbed delayed renewal process" W^ε (e.g., jump intensity ?)
- Regularity of V^ε ?
- $V^\varepsilon \to V$? In what sense ?

Potential Solution (ε-Optimal Strategy)

- First argue that the original problem is equivalent to one on $\tilde{D}$ (i.e., any solution on D must satisfy $V = 0$ on $\tilde{D} \setminus D$.
- Perturb domain $\tilde{D}$ (to D_θ such that $D_\theta \searrow \tilde{D}$), and denote the corresponding solution to the approximating HJB by $V^{\varepsilon,\theta}$
- Show that $V^{\varepsilon,\theta}$ is "*classical*" in *int* D_θ (whence on $\tilde{D}$!)
- Let $\theta \to 0$ and $\varepsilon \to 0$ (similar to the "x-continuity" before), ...

Note : The smoothness of $V^{\varepsilon,\theta}$ inside D_θ is possible because

- The PDE is now "non-degenerate" !
- **In progress :** Analysis on *Non-local HJB in bounded domains* (ref. Mou-Święch, Gong-Mou-Święch ('16, '17) is helping !)

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- We studied the optimal dividend/investment problem under a Sparre Andersen model assuming that the cumulative dividends has *bounded* rates (hence "regular").
- Using a *Backward Markovization techniques* we proved :
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THANK YOU VERY MUCH !