Multivariate Stress Testing for Solvency

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1. Introduction
   - Regulation
   - General Definition of Stress Test
   - Stressing Single Risk Factors
   - Reverse Stress Tests

Limitations of Stress Testing in Crisis

> *. . . weaknesses in infrastructure limited the ability of banks to identify and aggregate exposures across the bank. This weakness limits the effectiveness of risk management tools - including stress testing. . . . Prior to the crisis, most banks did not perform stress tests that took a comprehensive firm-wide perspective across risks and different books*

Methodological shortcomings

- Shocking single parameters/risk factors. Limited!
- Shocking many risk factors simultaneously. How?
- Using historical events. Not able to capture risks in new products; not severe enough
- Using hypthetical stress tests. Guessing! Prior to crisis difficult to obtain senior management buy-in for more extreme scenarios.

FSA Proposes Reverse Stress Test

> *We are proposing to introduce a 'reverse-stress test' requirement, which would apply to banks, building societies, CRD investment firms and insurers, and would require firms to consider the scenarios most likely to cause their current business model to become unviable.*
> http://www.fsa.gov.uk/pubs/cp/cp08_24_newsletter.pdf

How exactly is a reverse stress test to be constructed. And how does it differ from a standard (forward) stress test?

Aims of Presentation

1. To define a stress test and relate stress tests to the theory of risk measures.
2. To define a reverse stress test.
3. To discuss the construction of multivariate scenario sets for stress testing.
4. To give an overview of the elegant theory of stress testing for linear portfolios (which underlies a lot of standard procedures). This section draws on work by [McNeil and Smith, 2010] and unpublished material in second edition of [McNeil et al., 2005].

Set-up

- We fix a probability space $(\Omega, \mathcal{F}, P)$ and a set of financial risks $\mathcal{M}$ defined on this space. These risks are interpreted as portfolio or position losses over some fixed time horizon.
- We assume that $\mathcal{M}$ is a linear space containing constants, so that if $L_1, L_2 \in \mathcal{M}$, $m \in \mathbb{R}$ and $k > 0$ then $L_1 + L_2, L_1 + m, kL_1 \in \mathcal{M}$.
- A risk measure is a mapping $\varrho : \mathcal{M} \to \mathbb{R}$ with the interpretation that $\varrho(L)$ gives the amount of equity capital that is needed to back a position with loss L.
- A stress test is considered as an example of a risk measure.

General Definition

For a particular portfolio loss $L \in \mathcal{M}$ a stress test is carried out by computing

$$\varrho(L) = \sup \{L(\omega) : \omega \in S\}$$

for some subset $S \subset \Omega$.

- We consider a set of possible scenarios S that could take place over the time horizon and work out what the worst loss could be under these scenarios. This might be used to set capital.
- We also want to identify a ω_0 such that $L(\omega_0) = \varrho(L)$. (The sup will usually be a max.) The scenario ω_0 is sometimes called the least solvent likely event (LSLE).
- Probabilistic considerations enter in the choice of S.

Scenarios Based on Risk Factors

Typically losses will be related to a d-dimensional random vector of risk factors $\mathbf{X}$ (equity, interest-rate, FX, spread movements, etc.) by $L = \ell(\mathbf{X})$. Let $\Omega = \mathbb{R}^d$ and let each $\mathbf{x} \in \Omega$ represent a scenario for changes in these risk factors over the time period. The stress test is then

$$\varrho(L) = \sup\{\ell(\mathbf{x}) : \mathbf{x} \in S\}$$

for some subset of scenarios $S \subset \mathbb{R}^d$. Possibilities include

- A set of point scenarios $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$.
- An ellipsoidal scenario set $S = \{\mathbf{x} : (\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leq k\}$ for some parameters μ, Σ and k.

The latter corresponds to Studer's maximum loss concept.
[Studer, 1997, Studer, 1999, Breuer et al., 2009, Breuer et al., 2010]

Stress Tests as Risk Measures

- Stress tests are special cases of a class of risk measures known as generalized scenarios. These risk measures take the form

$$\varrho(L) = \sup \{ E_Q(L) : Q \in \mathcal{Q} \}$$

  where $\mathcal{Q}$ is a set of probability measures. In the stress test $\mathcal{Q}$ is a set of Dirac measures $\{\delta_{\mathbf{x}} : \mathbf{x} \in S\}$ which place all the probability on each scenario in S in turn.
- Generalized scenarios are coherent measures of risk. (Under some technical conditions all coherent measures of risk can be shown to be generalized scenarios.)
- In particular situations we can represent well known coherent risk measures as stress tests, as will later be seen.

Linear Portfolios

In reality the losses are likely to be non-linear functions of risk factors due to the presence of derivative-like assets (and liabilities) in a typical bank or insurance portfolio. But it is useful to consider portfolios with a linear dependence on risk factors for a number of reasons.

- Linear (delta) approximations are commonly applied in bank risk management.
- Some standard approaches are justified only under linear assumptions.

We will often consider linear portfolio losses in the set

$$\mathcal{M} = \left\{ L : L = m + \boldsymbol{\lambda}'\mathbf{X},\ m \in \mathbb{R}, \boldsymbol{\lambda} \in \mathbb{R}^d \right\}. \tag{1}$$

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How to Aggregate Single Factor Stresses?

It is common practice to stress risk factors one at a time. For example one might consider the impact in isolation of an equity market shock of x%, or a shift of y basis points in the yield curve, or a z% spike in the default rate of loans.

- How can we aggregate the resulting losses to take into account dependencies in these scenarios?
- Should we simply add them up?
- Should we overlay correlation assumptions? (Solvency II standard formula.)

A Possible Aggregation Formula

1. The d risk factors are stressed one at a time, in isolation, by pre-determined amounts $k_1, \ldots, k_d \in \mathbb{R}$.
2. Let $L = \ell(\mathbf{X})$. The corresponding losses relative to baseline $\Delta L_i = \ell(k_i \mathbf{e}_i) - \ell(\mathbf{0})$, $i = 1, \ldots, d$, are computed, where $\mathbf{e}_i$ are unit vectors and we assume $\Delta L_i > 0$.
3. The overall stress test is computed using the formula

$$\varrho(L) = \ell(\mathbf{0}) + \sqrt{\sum_{i=1}^{d} \sum_{j=1}^{d} \rho_{ij} \Delta L_i \Delta L_j}\,,$$

   where the ρ_{ij} are a set of correlation parameters forming elements of a symmetric matrix. Summation is a special case when the $\rho_{ij} = 1$ for all i and j.

When Is This Principles Based ?

When does this coincide with a proper stress test $\varrho(L) = \sup\{\ell(\mathbf{x}) : \mathbf{x} \in C\}$ for some appropriate set of multivariate scenarios C?

Theorem

Let $\mathcal{M}$ be the linear portfolio space in (1) and let $L = \ell(\mathbf{X}) = m + \boldsymbol{\lambda}'\mathbf{X} \in \mathcal{M}$. Let $\varrho(L)$ be given by the aggregation procedure described on previous slide. Provided the matrix $P = (\rho_{ij} sgn(k_i) sign(k_j))$ is positive-definite, we can write $\varrho(L) = \sup\{\ell(\mathbf{x}) : \mathbf{x} \in C\}$ where C is the ellipsoidal set

$$C = \left\{\mathbf{x} : \mathbf{x}'\Sigma^{-1}\mathbf{x} \leq 1\right\} ,$$

specified by $\Sigma = diag(\sigma_1, \ldots, \sigma_d) P diag(\sigma_1, \ldots, \sigma_d)$ where $\sigma_i = |k_i|$.

What Justifies Ellipsoidal Scenario Sets?

- We can interpret the correlation-adjusted summation rule as a computational recipe for a stress test when portfolios are linear and the scenario set is ellipsoidal.
- An ellipsoidal scenario set can be justified by the assumption of a multivariate normal distribution or an elliptical distribution for the risk factors.
- For elliptical distributions the contours of equal density are ellipsoids. The depth sets are also ellipsoidal as we will see in the next section.
- Both the elliptical and linear assumptions are strong assumptions which are unlikely to hold in practice.
- The summation rule, which appears conservative, can actually understate the risk in the presence of non-linearities.

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Reverse Stress Tests

Recall that in a standard forward stress test we want to compute

$$\varrho(L) = \sup\{\ell(\mathbf{x}) : \mathbf{x} \in S\}$$

for some subset of scenarios $S \subset \mathbb{R}^d$ and we want to identify the scenario that is responsible for the worst case loss. If S is closed this means

$$\mathbf{x}_{\text{LSLE}} = \arg\max\{\ell(\mathbf{x}) : \mathbf{x} \in S\}$$

In a reverse stress test we restrict attention to ruin scenarios

$$R = \left\{\mathbf{x} \in \mathbb{R}^d : \ell(\mathbf{x}) > 0\right\} .$$

We want to know the most plausible ways of being ruined, that is the scenarios in R that are most “probable” or “likely”. For continuous distributions on $\mathbb{R}^d$ this could be measured in terms of density, or another concept known as depth.

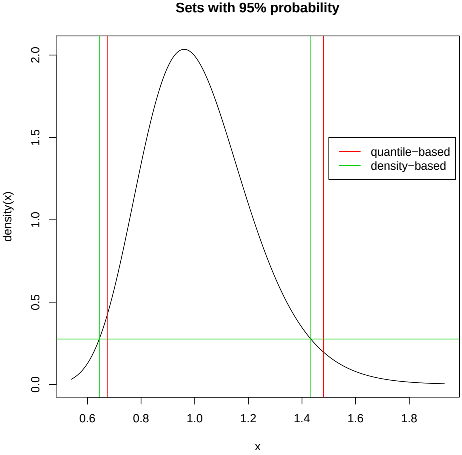
2 Constructing Multivariate Scenarios

- What are Plausible/Likely Scenarios?
- Quantiles in Higher Dimensions

In One Dimension

- For a single risk factor X we can use an inter-quantile range to define a set of plausible scenarios, particularly when X has a well-behaved unimodal distribution.
- For $0 < \theta < 1$ let $q_\theta(X)$ denote the θ-quantile of X. Assume that X has a continuous and strictly increasing distribution function so that $q_\theta(X)$ is always unambiguously defined.
- For any α satisfying $1 > \alpha > 0.5$, the inter-quantile interval $I = [q_{1-\alpha}(X), q_\alpha(X)]$ forms a set satisfying $P(I) = 2\alpha - 1$. For large α it is very likely that X will fall in this range.
- This is not the only interval with probability $2\alpha - 1$. Can also create sets which maximise the minimum density.

An Example


Sets with 95% probability
density(x)
x
quantile−based
density−based
0.0
0.5
1.0
1.5
2.0
0.6
0.8
1.0
1.2
1.4
1.6
1.8

2 Constructing Multivariate Scenarios
- What are Plausible/Likely Scenarios?
- Quantiles in Higher Dimensions

Notation for Higher-Dimensional Case

- For any point $\mathbf{y} \in \mathbb{R}^d$ and any directional vector $\mathbf{u} \in \mathbb{R}^d \setminus \{0\}$, consider the closed half space

$$H_{\mathbf{y},\mathbf{u}} = \left\{\mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{u}'\mathbf{x} \leq \mathbf{u}'\mathbf{y}\right\},$$

  bounded by the hyperplane through $\mathbf{y}$ with normal vector $\mathbf{u}$.
- The probability of the half-space is written

$$P_{\mathbf{X}}(H_{\mathbf{y},\mathbf{u}}) = P(\mathbf{u}'\mathbf{X} \leq \mathbf{u}'\mathbf{y}).$$

- We define an α-quantile function on $\mathbb{R}^d \setminus \{0\}$ by writing $q_\alpha(\mathbf{u})$ for the α-quantile of the random variable $\mathbf{u}'\mathbf{X}$.

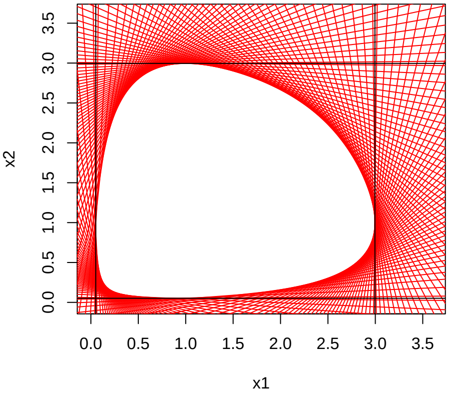
The Scenario Set

Let $\alpha > 0.5$ be fixed. We write our scenario set in two ways:

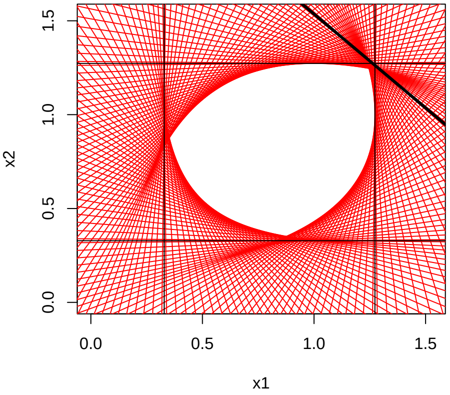
1. $$Q_\alpha = \bigcap \{H_{\mathbf{y},\mathbf{u}} \, : \, P_{\mathbf{X}}(H_{\mathbf{y},\mathbf{u}}) \geq \alpha\} \, ,$$
   the intersection of all closed half spaces with probability at least α;
2. $$Q_\alpha = \{\mathbf{x} \, : \, \mathbf{u}'\mathbf{x} \leq q_\alpha(\mathbf{u}) \, , \forall \mathbf{u}\} \, , \qquad (2)$$
   the set of points for which linear combinations are no larger than the quantile function.

The set Q_α is sometime referred to as a depth set consisting of points that are at least $1 - \alpha$ deep into the distribution.

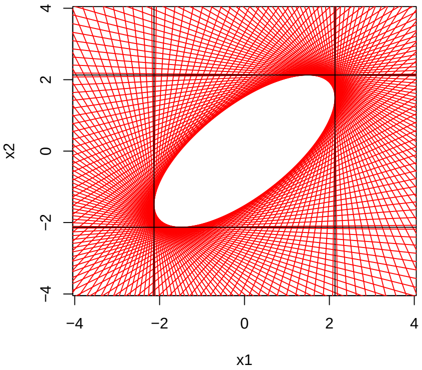
Two Independent Exponentials, $Q_{0.95}$



Two Independent Exponentials, $Q_{0.75}$



A bivariate Student distribution, $Q_{0.95}$



$\nu = 4$, $\rho = 0.7$

Commentary on examples

- Note how the depth set in the exponential case has a smooth boundary for $\alpha = 0.95$. (Supporting hyperplanes in every direction.)
- Note how the depth set in the exponential case has a sharp corners for $\alpha = 0.75$. (No supporting hyperplanes in some directions.)
- The depth set for an elliptical distribution is an ellipsoid.
- For elliptical distributions both the contours of equal depth and the contours of equal density are ellipsoidal.

3 Risk Measure Theory for Linear Portfolios

- Stress Test Representations for Standard Risk Measures
- Value-at-Risk
- Expected Shortfall
- Reverse Stress Tests

Coherent Risk Measures as Scenarios

Recall the definition of the linear portfolio set

$$\mathcal{M} = \left\{ L : L = m + \boldsymbol{\lambda}'\mathbf{X},\ m \in \mathbb{R}, \boldsymbol{\lambda} \in \mathbb{R}^d \right\}$$

and recall that a risk measure ϱ is coherent if it satisfies the following axioms:

Monotonicity. $L_1 \leq L_2 \Rightarrow \varrho(L_1) \leq \varrho(L_2)$.

Translation invariance. For $m \in \mathbb{R}$, $\varrho(L + m) = \varrho(L) + m$.

Subadditivity. For $L_1, L_2 \in \mathcal{M}$, $\varrho(L_1 + L_2) \leq \varrho(L_1) + \varrho(L_2)$.

Positive homogeneity. For $\lambda \geq 0$, $\varrho(\lambda x) = \lambda \varrho(x)$.

Duality Result

Theorem

A risk measure ϱ on the linear portfolio set $\mathcal{M}$ is coherent if and only if it has the stress test representation

$$\varrho(L) = \varrho(m + \boldsymbol{\lambda}'\mathbf{X}) = \sup\{m + \boldsymbol{\lambda}'\mathbf{x} : \mathbf{x} \in S_\varrho\}$$

where S_ϱ is the scenario set

$$S_\varrho = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{u}'\mathbf{x} \leq \varrho(\mathbf{u}'\mathbf{X}), \forall \mathbf{u} \in \mathbb{R}^d\}\,.$$

The scenario set is a closed convex set and we may conclude that

$$\varrho(L) = \varrho(m + \boldsymbol{\lambda}'\mathbf{X}) = m + \boldsymbol{\lambda}'\mathbf{x}_{LSLE}\,. \qquad (3)$$

The Case of VaR

Let us suppose the risk measure $\varrho = \mathrm{VaR}_\alpha$ for some value $\alpha > 0.5$. Then the scenario set S_ϱ is as given in (2), i.e.

$$\{\mathbf{x} \in \mathbb{R}^d : \mathbf{u}'\mathbf{x} \leq q_\alpha(\mathbf{u}), \forall \mathbf{u} \in \mathbb{R}^d\} = Q_\alpha \, .$$

But when is VaR_α a coherent risk measure?

Theorem

Suppose that $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ *(an elliptical distribution centred at* $\boldsymbol{\mu}$ *with dispersion matrix* Σ *and type* ψ*) and let* $\mathcal{M}$ *be the space of linear portfolios. Then* VaR_α *is coherent on* $\mathcal{M}$ *for* $\alpha > 0.5$.

The Case of VaR for Elliptical Distributions

- In the elliptical case the scenario set is

$$Q_\alpha = \{\mathbf{x} : (\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leq k_\alpha^2\}$$

  where $k_\alpha = \mathrm{VaR}_\alpha(Y)$ and $Y \sim E_1(0, 1, \psi)$.
- Moreover the LSLE can be calculated by the method of Lagrange multipliers and is

$$\mathbf{x}_{\mathrm{LSLE}} = \boldsymbol{\mu} + \frac{\Sigma\boldsymbol{\lambda}}{\sqrt{\boldsymbol{\lambda}'\Sigma\boldsymbol{\lambda}}} k_\alpha.$$

- The corresponding stress loss is

$$\mathrm{VaR}_\alpha(m + \boldsymbol{\lambda}'\mathbf{X}) = m + \boldsymbol{\lambda}'\mathbf{x}_{\mathrm{LSLE}} = m + \boldsymbol{\lambda}'\boldsymbol{\mu} + \sqrt{\boldsymbol{\lambda}'\Sigma\boldsymbol{\lambda}}\, k_\alpha.$$

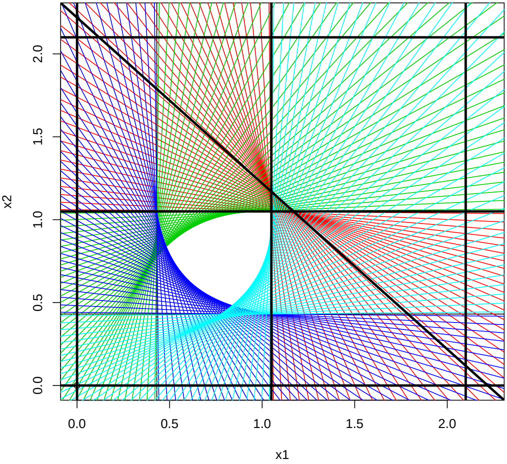
The Case of VaR for Non-Elliptical Distributions

- In the non-elliptical case it may happen that VaR_α is not coherent on $\mathcal{M}$ for some value of α. In such situations we may find portfolio weights $\boldsymbol{\lambda}$ such that

$$\text{VaR}_\alpha(L) = \text{VaR}_\alpha(m + \boldsymbol{\lambda}'\mathbf{X}) > \sup\left\{m + \boldsymbol{\lambda}'\mathbf{x} : \mathbf{x} \in Q_\alpha\right\} .$$

- Such a situation was shown earlier. It occurs when some lines bounding half-spaces with probablity α are not supporting hyperplanes for the set Q_α, i.e. they do not touch it.
- In such situations we can construct explicit examples to show that VaR_α violates the property of subadditivity.

Two Independent Exponentials, $Q_{0.65}$



Demonstration of Super-Additivity

- In previous slide we set $\alpha = 0.65$ and consider loss $L = X_1 + X_2$.
- Diagonal line is $x_1 + x_2 = q_\alpha(X_1 + X_2)$ which obviously intersects axes at $(0, q_\alpha(X_1 + X_2))$ and $(q_\alpha(X_1 + X_2), 0)$.
- Horizontal (vertical) lines are at 0, $q_\alpha(X_1)$ and $2q_\alpha(X_1)$.
- We infer
  1. $x_1 + x_2 < q_\alpha(X_1 + X_2)$ in the depth set;
  2. $\sup\{x_1 + x_2 : \mathbf{x} \in Q_\alpha\}$ is a poor lower bound
  3. $q_\alpha(X_1 + X_2) > q_\alpha(X_1) + q_\alpha(X_2)$ (non-subadditivity of quantile risk measure)

3. Risk Measure Theory for Linear Portfolios
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The Case of Expected Shortfall

Consider the expected shortfall risk measure $\varrho = \mathrm{ES}_\alpha$, which is known to be a coherent risk measure given by

$$\mathrm{ES}_\alpha(L) = \frac{\int_\alpha^1 \mathrm{VaR}_\theta(L)d\theta}{1-\alpha}, \quad \alpha \in (0.5, 1),$$

and write $e_\alpha(\mathbf{u}) := \mathrm{ES}_\alpha(\mathbf{u}'\mathbf{X})$.
Since expected shortfall is a coherent risk measure (irrespective of $\mathbf{X}$) it must have the stress test representation

$$\mathrm{ES}_\alpha(L) = \varrho(m + \boldsymbol{\lambda}'\mathbf{X}) = \sup\{m + \boldsymbol{\lambda}'\mathbf{x} : \mathbf{x} \in E_\alpha\}$$

where

$$E_\alpha := \{\mathbf{x} \ : \ \mathbf{u}'\mathbf{x} \leq e_\alpha(\mathbf{u}), \forall \mathbf{u}\}.$$

The Case of ES for Elliptical Distributions

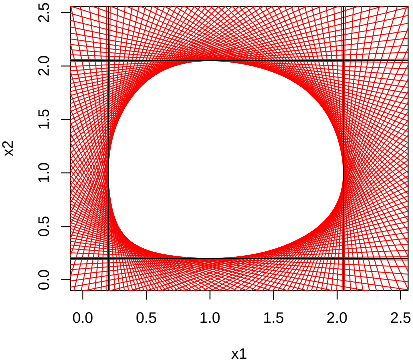
If $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ is elliptically distributed then the scenario set is simply the ellipsoidal set

$$E_\alpha = \{\mathbf{x} \ : \ (\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leq l_\alpha^2\},$$

where $l_\alpha = \mathrm{ES}_\alpha(Y)$ and $Y \sim E_1(0, 1, \psi)$.
The LSLE is given by

$$\mathbf{x}_{\mathrm{LSLE}} = \boldsymbol{\mu} + \frac{\Sigma\boldsymbol{\lambda}}{\sqrt{\boldsymbol{\lambda}'\Sigma\boldsymbol{\lambda}}}\, l_\alpha \ .$$

The Case of ES for Non-Elliptical Distributions



The set $E_{0.65}$. Recall that $Q_{0.65}$ did not have smooth boundary.

Most likely ruin event

We use depth as a measure of plausibility and define it to be

$$\mathrm{depth}(\mathbf{x}) = \sup\left\{\theta : \mathbf{x} \in Q_{1-\theta}\right\} ,$$

the largest θ for which $\mathbf{x}$ is in the depth set $Q_{1-\theta}$. The most likely ruin event (MLRE) for linear portfolios will be

$$\mathbf{x}_{\mathrm{MLRE}} = \arg\max\left\{\mathrm{depth}(\mathbf{x}) : m + \boldsymbol{\lambda}'\mathbf{x} \geq 0\right\} .$$

Elliptical Case

We have a simple optimization to solve:

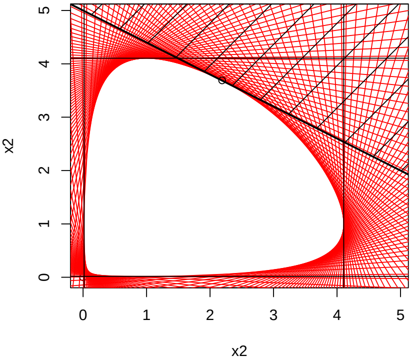
$$\mathbf{x}_{\mathrm{MLRE}} = \arg\min\{(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) \ : \ m + \boldsymbol{\lambda}'\mathbf{x} \geq 0\}\,.$$

This has the solution:

$$\mathbf{x}_{\mathrm{MLRE}} = \boldsymbol{\mu} - \frac{\Sigma\boldsymbol{\lambda}}{\boldsymbol{\lambda}'\Sigma\boldsymbol{\lambda}}\left(m + \boldsymbol{\lambda}'\boldsymbol{\mu}\right)\,.$$

Non-Elliptical Case: Two Exponentials

In the example we set $\ell(\mathbf{x}) = 3x_1 + 5x_2 - 25$.



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