Risk-Minimization with Incomplete Information in a Model for High-Frequency Data*

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Abstract

We study risk-minimizing hedging-strategies for derivatives in a model where the asset price follows a marked point process with stochastic jump-intensity, which depends on some unobservable state-variable process. This model reflects stylized facts which are typical for high frequency data. We assume that agents in our model are restricted to observing past asset prices. This poses some problems for the computation of risk-minimizing hedging strategies as the current value of the state variable is unobservable for our agents. We overcome this difficulty by a two-step procedure, which is based on a projection-result of Schweizer and show that in our context the computation of risk-minimizing strategies leads to a filtering problem which has received some attention in the nonlinear filtering literature.

Key words: Incomplete Markets, Risk-Minimizing Hedging Strategies, High-Frequency Data, Marked Point Processes
JEL classification: G12, G13

1 Introduction

With the availability of more and better data a considerable amount of recent research in finance has been devoted to the analysis and the modelling of high-frequency data (tick by tick data) for asset prices. Here we mention only the pioneering work of the Olson-group; see for instance Guillaume, Dacorogna, Davé, Müller, Olsen, and Pictet (1997). It was realized quickly that diffusion models which are routinely used in derivative asset analysis are of limited use in modelling tick data, as on a very small time scale the behaviour of real asset returns is markedly different from the behaviour of diffusion processes. In particular, real asset prices are piecewise constant and jump only at discrete points in time, e.g. in reaction to trades or to significant new information. Diffusion processes to the contrary have continuous trajectories with nonzero quadratic variation.

Here we propose an alternative model, where asset prices are given by a marked point process. The jump-intensity of this process is assumed to depend on some exogeneous state variable which is not directly observable for the agents in our model. This models the fact that real markets exhibit random fluctuations of market activity, which are related among others to fluctuations in the amount of incoming news. Similar models have been proposed

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among others by Rogers and Zane (1998), Rydberg and Shephard (1999) and by Frey and Runggaldier (1999a). These models are close in spirit to models for aggregated claims in actuarial risk theory; see for instance Rolski, Schmidli, Schmidt, and Teugels (1999).

In the present paper we are mainly concerned with the hedging of derivatives in the context of our marked point process model. Since our market is incomplete due to the presence of jumps and stochastic jump-intensity, we have to choose some approach to hedging derivatives under incompleteness to determine hedging strategies. Here we use the criterion of risk minimization proposed by Föllmer and Sondermann (1986).

In contrast to most of the previous literature we assume in the present paper, that the hedger has only access to the information contained in past asset prices. While this assumption is perfectly realistic from an economic viewpoint, it causes difficulties for the computation of hedge strategies as the current value of the jump-intensity is not directly observable for our agent. We therefore use a two-step approach to computing our hedge-strategy. In the first step we compute a risk-minimizing strategy for a ficticious agent who is informed about the jump-intensity (full-information case). In a second step we use a projection result developed by Schweizer (1994) to compute a risk-minimizing strategy for an agent who is restricted to observing the asset price process (incomplete information case). It will be shown that for path-independent claims the computation of risk-minimizing strategies under incomplete information leads to an interesting nonlinear filtering problem which can be solved by well-known approximation techniques.

The computation of risk-minimizing hedge-strategies with incomplete information has also been considered in the related paper Frey and Runggaldier (1999b). The present paper differs from Frey and Runggaldier (1999b) in two important respects. First, the model for the asset price is different. In Frey and Runggaldier (1999b) it is assumed that the asset price follows a standard stochastic volatility model with continuous trajectories. Hence standard techniques can be employed to compute risk-minimizing hedge strategies under full information, whereas in the present paper, where the asset price follows a marked point process, we develop a new – albeit straightforward – approach to deal with the full-information-case. Moreover, the information available to the hedger is different in both models. In Frey and Runggaldier (1999b) the incomplete-information situation arises, as we assume that the hedger is unable to monitor the asset continuously but is confined to observations at discrete random points in time. In the present paper we assume – perhaps more naturally – that the hedger has access to price information on a continuous basis; the incomplete-information situation arises, as in our model information about past prices alone is not sufficient to determine the current value of the state variable process.

2 The Model

We are working in a stylized market with two traded assets: a riskless money market account and a risky asset referred to as the stock. For simplicity we take the money market account as numeraire. The price of the stock accounted in units of the numeraire is modeled as a stochastic process $(S_t)_{0\le t\le \overline{T}}$ on some filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ for $0 \le t \le \overline{T}$. Here $\overline{T}$ denotes some finite horizon at which the model ends. We assume that the price of the stock changes only at discrete, random points in time $T_1 < T_2 < \ldots \le \overline{T}$. The T_n represent instants at which a large trade occurs or at which a market maker updates his quotes in reaction to significant new information. More precisely, we model the asset price as exponential martingale of some doubly stochastic Poisson process R (the return-process), i.e. we take S as solution to the SDE

$$dS_t = S_{t-} dR_t\,, \text{ where } R_t = \sum_{i=n}^{N_t} Z_n \tag{2.1}$$

for a doubly stochastic Poisson process N (also referred to as Cox-process) and a iid-sequence of random variables (rv's) $(Z_n)_n$, which model the percentage return at the n-th event. R and S are $\{\mathcal{F}_t\}$-adapted.

We now give a formal definition of our return process. Consider some adapted, time homogeneous Markov-process X on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ (the driving process for N) and a continuous function $\lambda : [0, \overline{T}] \times \mathbb{R} \to \mathbb{R}$ taking its values in the interval $[\underline{\lambda}, \overline{\lambda}]$ for constants $\underline{\lambda}$ and $\overline{\lambda}$ with $0 < \underline{\lambda} \leq \overline{\lambda} < \infty$. Besides $\{\mathcal{F}_t\}$ we will also consider the filtrations $\{\mathcal{F}_t^S\}$ and $\{\mathcal{F}_t^N\}$ generated by the processes S and N as well as the filtration $\{\mathcal{G}_t\}$ defined via

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(X_s,\, 0 \leq s \leq \overline{T})\,.$$

Note that $\mathcal{G}_0$ contains information about all the future of the process X. We make

A1) N is a Cox-process with driving process X and intensity function $\lambda(t, X_t)$, i.e. N is a point-process which admits the $(P, \{\mathcal{G}_t\})$-intensity $\lambda(t, X_t)$. In particular, $M_t^N := N_t - \int_0^t \lambda(s, X_s) ds$ is a $(P, \{\mathcal{G}_t\})$-martingale.

Remark 2.1. Assumption A1 implies that, given information about all the future of the state variable, N is a Poisson process with conditionally deterministic intensity $\lambda(t, X_t)$. This is in general a stronger requirement than the assumption that N is a point process with $(P, \{\mathcal{F}_t\})$-intensity $\lambda(t, X_t)$. We refer the reader to the book Brémaud (1981) for a discussion of this point and for a detailed account of marked point processes.

Remark 2.2. A possible economic interpretation of the quantities introduced in A1 is as follows: $\lambda(t, X_t)$ corresponds to the rate at which new economic information is absorbed by the agents active on the market for the stock; it makes perfect sense to assume that this rate is influenced by some exogeneous stochastic factor X. The time-dependency in λ is introduced in order to incorporate seasonality effects, which are typical for high frequency data; see e.g. Guillaume, Dacorogna, Davé, Müller, Olsen, and Pictet (1997).

In the present paper we are mainly concerned with the case where X follows a diffusion.

A2) (X is a diffusion) Consider a Wiener process $w = (w_t)_{0 \leq t \leq \overline{T}}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and continuous functions α and β: $\mathbb{R} \to \mathbb{R}$. X is a global solution to the SDE

$$dX_t = \alpha(X_t) dt + \beta(X_t) dw_t\,. \tag{2.2}$$

Moreover, weak uniqueness holds for the SDE (2.2).

Finally we make the following assumption on the distribution of the rv's Z_n.

A3) $(Z_n)_n$ is a sequence of independent, identically ν-distributed rv's which are moreover independent of N and X. We have

$$\nu[(-1, \infty)] = 1\,, \int_{-\infty}^{\infty} z\nu(dz) = 0\,, \text{ and } \sigma^2 := \int_{-\infty}^{\infty} z^2 \nu(dz) < \infty\,. \tag{2.3}$$

We now discuss some properties of the processes S and R.

EXPLICIT FORMULA FOR S: Using the exponential formula from Lebesgue-Stieltjes calculus it is immediately seen that S_t can be written as

$$S_t = S_0 \exp\left(\sum_{n=1}^{N_t} \log(1+Z_n)\right) ; \tag{2.4}$$

in particular, the log-price process is given by a doubly stochastic compound Poisson process with marks $Y_n := \log(1+Z_n)$.

COUNTING MEASURE AND SEMIMARTINGALE DECOMPOSITION: By p^R we denote the counting-measure associated with the process R; p^R is a random measure on $[0,\overline{T}] \times \mathbb{R}$ such that for functions $W : \Omega \times [0,\overline{T}] \times \mathbb{R} \to \mathbb{R}$

$$\int_0^t \int_{-\infty}^{\infty} W(\omega; s, z) p^R(ds \times dz) = \sum_{n=1}^{\infty} W\left(\omega; T_n(\omega), Z_n(\omega)\right) 1_{\{T_n(\omega) \le t\}} . \tag{2.5}$$

By definition R has the $(P, \{\mathcal{F}_t\})$-local characteristics $(\lambda(t, X_t), \nu(dz))$, i.e. defining the signed random measure

$$q^R(dt \times dz) = p^R(dt \times dz) - \nu(dz)\lambda(t, X_t)dt , \tag{2.6}$$

we have for every predictable function $W : \Omega \times [0,\overline{T}] \times \mathbb{R} \to \mathbb{R}$ such that the integral $\int_0^{\overline{T}} \int_{-\infty}^{\infty} |W(\omega; t, z)| \nu(dz)\lambda(t, X_t)dt$ is finite P a.s. that

$$\int_0^t \int_{-\infty}^{\infty} W(\omega; s, z) q^R(ds \times dz) \text{ is a } P\text{-local martingale;} \tag{2.7}$$

see Chapter VIII of Brémaud (1981) for details. Now we obviously have

$$R_t = \int_0^t \int_{-\infty}^{\infty} z\, p^R(ds \times dz) = \int_0^t \int_{-\infty}^{\infty} z\, q^R(ds \times dz) ,$$

as $\int_{-\infty}^{\infty} z\, \nu(dz) = 0$. Hence under Assumption A3 the return process R and the stock price process S are P-local martingales, i.e. P is already a risk-neutral measure.

QUADRATIC VARIATION OF R AND S: We get from Itô's formula for general semimartingales and the fact that the R is a quadratic pure jump process (see e.g. Protter (1992)) that

$$R_t^2 = 2 \int_0^t R_{s-} dR_s + \sum_{n=1}^{N_t} (Z_n)^2 .$$

Now the stochastic integral on the rhs is a local martingale. Moreover,

$$\sum_{n=1}^{N_t} (Z_n)^2 = \int_0^t \int_{-\infty}^{\infty} z^2 q^R(ds \times dz) + \int_0^t \sigma^2 \lambda(s, X_s) ds .$$

Hence $R_t^2 - \int_0^t \sigma^2 \lambda(s, X_s) ds$ is a P-local martingale, and we have

$$\langle R \rangle_t = \int_0^t \sigma^2 \lambda(s, X_s) ds \quad \text{and} \tag{2.8}$$

$$\langle S \rangle_t = \int_0^t S_{s-}^2 \sigma^2 \lambda(s, X_s) ds . \tag{2.9}$$

SQUARE INTEGRABILITY: Define for $0 \le t < T \le \overline{T}$ the random variable $\Lambda(t,T)$ by

$$\Lambda(t,T) := \int_t^T \lambda(s, X_s)ds \,. \tag{2.10}$$

Lemma 2.3. *Under A1 and A3 the asset price process is a square-integrable martingale; in particular,* $E[S_t^2] = S_0^2 E[\exp(\sigma^2 \Lambda(0,t))]$.

PROOF: We get from (2.4) that $S_t^2 = S_0^2 \exp\left(\sum_{n=1}^{N_t} 2\log(1+Z_n)\right)$. Recall the definition of the filtration $\{\mathcal{G}_t\}$ before A1. We have $E[S_t^2] = E\left[E[S_t^2|\mathcal{G}_0]\right]$, and

$$E[S_t^2|\mathcal{G}_0] = S_0^2 \sum_{n=0}^{\infty} E\left[\exp\left(2\sum_{i=1}^{n}\log(1+Z_n)\right)\right] P[N_t = n|\mathcal{G}_0]\,.$$

Using that the $(Z_n)_n$ form an iid-sequence we get

$$E\left[\exp\left(2\sum_{i=1}^{n}\log(1+Z_n)\right)\right] = \prod_{i=1}^{n} E[(1+Z)^2] = (1+\sigma^2)^n\,.$$

Hence

$$E[S_t^2|\mathcal{G}_0] = S_0^2 \sum_{n=0}^{\infty} \frac{(1+\sigma^2)^n \Lambda(0,T)^n}{n!\exp(\Lambda(0,T))} = S_0^2 \exp(\sigma^2\Lambda(0,t))\,.$$

This yields $E[S_t^2] = S_0^2 E[\exp(\sigma^2\Lambda(0,t))]$, and in particular $E[S_t^2] \le S_0^2 \exp(\sigma^2\,\overline{\lambda}\,\overline{T}) < infty$. Together with the boundedness of λ this implies that

$$E[\langle S\rangle_t] = E\left[\int_0^t S_{s-}^2 \sigma^2 \lambda(s,X_s)ds\right] \le \overline{\lambda}\sigma^2 \int_0^t E[S_s^2]ds < \infty,$$

which proves the Lemma. □

MARKOV-PROPERTY: It is easily seen that the pair (S, X) forms an inhomogeneous two-dimensional Markov process.

3 Risk-Minimizing Hedging Strategies

We now turn to the hedging of derivative securities in our framework. We assume that Assumptions A1, A2 and A3 are satisfied; hence the asset price is a square integrable martingale. Since our market is incomplete we have to choose some approach to hedging derivatives under incompleteness to determine hedging strategies. In this paper we shall use the criterion of risk minimization as analyzed in Föllmer and Sondermann (1986) and Föllmer and Schweizer (1991).

We consider the case where the hedger has only access to the information contained in past asset prices, i.e. to the filtration $\{\mathcal{F}_t^S\}$. While this assumption is perfectly realistic from an economic viewpoint, it causes difficulties for the computation of hedging strategies, as the driving process X is typically not $\{\mathcal{F}_t^S\}$-adapted. We therefore use a two-step approach to computing our hedge-strategy. In the first step we compute a risk-minimizing strategy for an agent who has access to the filtration $\{F_t\}$ (full information case). Here we draw heavily on the Markov-property of the $\{F_t\}$-adapted process (S, X). In a second step we use a projection result developed by Schweizer (1994) to compute a $\{\mathcal{F}_t^S\}$-risk-minimizing strategy. The martingale property of S is crucial here as the result of Schweizer is applicable only if the asset price process is a local martingale. Although the projection results for risk minimizing hedging strategies under restricted information hold for general claims $H \in \mathcal{L}^2(\Omega, \mathcal{F}_T^S, P)$, for actual computation we shall restrict ourselves to claims, whose payoff is a function $H(S_T)$ with $E\{(H(S_T))^2\} < \infty$. By Lemma 2.3 a European call option is a particular exaple of such a derivative. It will be shown that for these claims the computation of $\{\mathcal{F}_t^S\}$-risk-minimizing strategies leads to a nonlinear filtering problem which can be solved by well-known approximation techniques based on the reference-probability approach to nonlinear filtering as outlined for instance in Chapter VI of Brémaud (1981).

We are well aware that the choice of a particular martingale measure matters when studying risk-minimization in an incomplete markets context. However, in the present paper we are not interested in selecting a "good" martingale measure; rather we want to focus on the incomplete information problem, which arises as X is unobservable for our hedger. We therefore believe that in our context the martingale assumption is acceptable.

3.1 Risk-Minimization under Full Information

It is well-known from Föllmer and Sondermann (1986) that in our situation where the asset price is a P-martingale an $\{F_t\}$-risk-minimizing strategy is determined by the Kunita-Watanabe decomposition of the P-martingale (H_t) defined by $H_t = E[H(S_T)|\mathcal{F}_t]$ with respect to the P-martingale S. More precisely, the stock-position is given by the $\{F_t\}$-predictable integrand $\xi^{\mathcal{F}}$ of the stochastic integral with respect to S, and the value-process of the risk-minimizing strategy is given by the $\{F_t\}$-adapted process (H_t).

COMPUTATION OF H_t: By the Markov-property of the pair (S, X) it suffices to consider the case $t = 0$. Defining the function $\tilde{H}$ by $\tilde{H}(x) := H(\exp(x))$ we get from (2.4)

$$E[H(S_T)] = E[H(S_T)|\mathcal{G}_0] = E\left[\tilde{H}\left(\log S_0 + \sum_{n=1}^{N_T} \log(1+Z_n)\right) \mid \mathcal{G}_0\right]. \tag{3.1}$$

Recall that under Assumption A1 the conditional distribution of N_T with respect to $\mathcal{G}_0$ is a Poisson-distribution with parameter $\Lambda(0,T)$. Moreover, under A3 the $(Z_n)_n$ are a sequence of independent and identically ν-distributed random variables independent of N and X, such that the random sum $\sum_{n=1}^{N_T} \log(1+Z_n)$ follows a compound Poisson distribution with characteristics $(\Lambda(0,T), \nu)$. Hence we may write

$$E\left[\tilde{H}\left(\log S_0 + \sum_{n=1}^{N_T} \log(1+Z_n)\right) \mid \mathcal{G}_0\right] = \tilde{h}(S_0, \Lambda(0,T))$$

for some function $\tilde{h}$: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, and (3.1) yields $E[H(S_T)] = E[\tilde{h}(S_0, \Lambda(0,T))]$. For actual computations we first have to compute the function $\tilde{h}$; afterwards we have to average the result over the distribution of $\Lambda(0,T)$. For both problems we can draw on well-established techniques:

- Compound Poisson distributions arise frequently in actuarial risk-theory and actuaries have developed efficient techniques for their evaluation, which can be applied in computing $\tilde{h}$. In particular, if the distribution of the jump-size $Y_n := \log(1+Z_n)$ is assumed to be discrete, we may use the well-known Panjer-Willmot recursion; see e.g. Panjer and Willmot (1992) or Chapter 4.4 of Rolski, Schmidli, Schmidt, and Teugels (1999).

- Analytical and numerical methods for computing the distribution of $\Lambda(0,T)$ have been developed in the literature on stochastic volatility models. We refer the reader to Ball and Roma (1994) for a survey.

MARTINGALE-REPRESENTATION OF (H_t): Define the function $h : [0,T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ by

$$h(t,S,x) := E_{(t,S,x)}[H(S_T)] \tag{3.2}$$

and note that $H_t = h(t, S_t, X_t)$ by the Markov-property of the pair (S,X). Assuming that h is sufficiently smooth we now derive a representation of the martingale (H_t) as stochastic integral with respect to the compensated counting measure q^R and the Brownian motion w driving the dynamics of X. As a by-product we obtain an integro-differential equation for the function h.

We have, provided that h is continuous, $\mathcal{C}^1$ in the first and $\mathcal{C}^2$ in the third argument,

$$\begin{aligned} h(t,S_t,X_t) = h(0,S_0,X_0) &+ \int_0^t h_x(s,S_{s-},X_s)\beta(X_s)dw_s \\ &+ \int_0^t \left(h_t(s,S_{s-},X_s) + \alpha(X_s)h_x(s,S_{s-},X_s) + \frac{1}{2}\beta^2(X_s)h_{xx}(s,S_{s-},X_s)\right) ds \\ &+ \int_0^t \int_{-\infty}^{\infty} \left(h(s,S_{s-}(1+z),X_s) - h(s,S_{s-},X_s)\right) p^R(ds \times dz)\,. \end{aligned} \tag{3.3}$$

Decomposing the integral with respect to p^R in (3.3) as

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} \left(h(s,S_{s-}(1+z),X_s) - h(s,S_{s-},X_s)\right) p^R(ds \times dz) \\ &\quad = \int_0^t \int_{-\infty}^{\infty} \left(h(s,S_{s-}(1+z),X_s) - h(s,S_{s-},X_s)\right) q^R(ds \times dz) \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \left(h(s,S_{s-}(1+z),X_s) - h(s,S_{s-},X_s)\right) \nu(dz)\lambda(s,X_s)ds\,, \end{aligned}$$

we get the semimartingale-decomposition of the process $h(t,S_t,X_t)$. Now $h(t,S_t,X_t)$ is a local martingale by definition, so that the predictable terms of finite variation in its semimartingale-decomposition (the integrals with respect to ds) must vanish. This yields the following integro-differential equation for h

$$\begin{aligned} &h_t(t,S,x) + \alpha(x)h_x(t,S,x) + \frac{1}{2}\beta^2(x)h_{xx}(t,S,x) + \\ &\qquad + \lambda(t,x)\int_{-\infty}^{\infty} \left(h(t,S(1+z),x) - h(s,S,x)\right)\nu(dz) = 0 \end{aligned} \tag{3.4}$$

$$h(T,S,x) = H(S)\,. \tag{3.5}$$

Note that in (3.5) we have used that $S_T = S_{T-}$ a.s. Moreover, we get the following martingale representation for (H_t)

$$\begin{aligned} H_t = H_0 &+ \int_0^t h_x(s,S_{s-},X_s)\beta(X_s)dw_s \\ &+ \int_0^t \int_{-\infty}^{\infty} \left(h(s,S_{s-}(1+z),X_s) - h(s,S_{s-},X_s)\right) q^R(ds \times dz)\,. \end{aligned} \tag{3.6}$$

Computation of the hedge-strategy $\xi^{\mathcal{F}}$: As shown for instance in Föllmer and Schweizer (1991), $\xi^{\mathcal{F}}$ can be computed as $\{\mathcal{F}_t\}$-predictable density of the quadratic co-variation of (H_t) and S with respect to the quadratic variation of S, i.e.

$$\xi^{\mathcal{F}} = \frac{d\langle H, S\rangle}{\langle S\rangle}\,. \tag{3.7}$$

The "brackets" in (3.7) are of course taken with respect to the filtration $\{\mathcal{F}_t\}$.

We now turn to computing $\langle H, S\rangle_t$. As $\langle H, S\rangle_t = \int_0^t S_{s-}d\langle H, R\rangle_s$, it suffices to determine $\langle H, R\rangle_t$. Now, R being a finite variation process and w being continuous, we have $\langle w, R\rangle = 0$. On the other hand it is well-known that for $\{\mathcal{F}_t\}$-predictable functions $g_1(t,\omega,z)$ and $g_2(t,\omega,z)$ we have

$$\Big\langle \int_0^{\cdot}\int_{-\infty}^{\infty} g_1(s,\omega,z)q^R(ds\times dz)\,,\ \int_0^{\cdot}\int_{-\infty}^{\infty} g_2(s,\omega,z)q^R(ds\times dz)\Big\rangle_t = \int_0^t\int_{-\infty}^{\infty} g_1(s,\omega,z)g_2(s,\omega,z)\,\nu(dz)\lambda(s,X_s)ds\,.$$

Hence we get

$$\langle H, R\rangle_t = \int_0^t\int_{-\infty}^{\infty}\left(h(s,S_{s-}(1+z),X_s) - h(s,S_{s-},X_s)\right)z\,\nu(dz)\lambda(s,X_s)ds\,. \tag{3.8}$$

The quadratic variation $\langle S\rangle$ is known from (2.9). Hence we get using (3.7)

$$\xi_t^{\mathcal{F}} = \frac{g(t,S_{t-},X_t)}{\sigma^2 S_{t-}}\,,\ \text{where } g(t,S,x) := \int_{-\infty}^{\infty}\left(h(t,S(1+z),x) - h(t,S,x)\right)z\,\nu(dz)\,. \tag{3.9}$$

It is apparent from (3.9) that $\xi_t^{\mathcal{F}}$ can be viewed as infinitesimal regression of ΔH_t on ΔS_t.

3.2 Risk-Minimization with respect to $\{\mathcal{F}_t^S\}$ and Nonlinear Filtering

We now consider the case where the hedger is constraint to observing past asset prices and compute an $\{\mathcal{F}_t^S\}$-risk-minimizing strategy. By this we mean a strategy with stockholdings $\xi^{(\mathcal{F}^S)}$ and value-process $H^{(\mathcal{F}^S)}$ which is $\{\mathcal{F}_t^S\}$-admissible (i.e. $\xi^{(\mathcal{F}^S)}$ is $\{\mathcal{F}_t^S\}$-predictable, $H^{(\mathcal{F}^S)}$ is $\{\mathcal{F}_t^S\}$-adapted and the gains from trade $\int_0^t \xi_s^{(\mathcal{F}^S)}dS_s$ are a square-integrable martingale) and which minimizes moreover the remaining risk in the sense of Föllmer and Sondermann (1986) in the class of all $\{\mathcal{F}_t^S\}$-admissible strategies. We have

Proposition 3.1. *Consider a square-integrable contingent claim with payoff $H(S_T)$. Then the $\{\mathcal{F}_t^S\}$-risk-minimizing hedging strategy with value-process $H^{(\mathcal{F}^S)}$ and stockholdings $\xi^{(\mathcal{F}^S)}$ is given by*

$$H_t^{(\mathcal{F}^S)} = E\left[h(t,S_t,X_t)|\mathcal{F}_t^S\right] \quad \text{and} \tag{3.10}$$

$$\xi_t^{(\mathcal{F}^S)} = S_{t-}^{-1}\sigma^{-2}\frac{\left(E[\lambda(t,X_t)g(t,S_{t-},X_t)|\mathcal{F}_{t-}^S]\right)^-}{\left(E[\lambda(t,X_t)|\mathcal{F}_{t-}^S]\right)^-}\,, \tag{3.11}$$

where the functions h and g are defined in (3.2) and (3.9), respectively.

Proof: We know from Föllmer and Sondermann (1986) that $H_t^{\mathcal{F}^S} = E\left[H(S_T)|\mathcal{F}_t^S\right]$. Moreover, we get from the law of iterated expectations

$$H_t^{(\mathcal{F}^S)} = E\left[[H(S_T)|\mathcal{F}_t]|\mathcal{F}_t^S\right] = E\left[h(t,S_t,X_t)|\mathcal{F}_t^S\right]\,, \tag{3.12}$$

which proves (3.10).

We now turn to the hedge strategy $\xi^{(\mathcal{F}^S)}$. We denote for any measurable stochastic process U on $(\Omega, \mathcal{F}, P)$ by ${}^{(o,\mathcal{F}^S)}U$ (respectively by ${}^{(p,\mathcal{F}^S)}U$) the optional projection (respectively the predictable projection) of U on the filtration $\{\mathcal{F}_t^S\}$; see e.g. Chapter VI of Rogers and Williams (1987) for a formal definition.

Denote by v the density of $d\langle S\rangle$ with respect to Lebesgue-measure, i.e. $v_t = \sigma^2 S_t^2 \lambda(t, X_t)$. It follows from Theorem 2.5 and in particular from equation (3.3) of Schweizer (1994) that

$$\xi_t^{(\mathcal{F}^S)} = \frac{{}^{(p,\mathcal{F}^S)}(v_t \xi_t^{\mathcal{F}})}{{}^{(p,\mathcal{F}^S)}(v_t)} . \tag{3.13}$$

It remains to determine the predictable projections in (3.13). Define for a continuous function f: $[0, \overline{T}] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ the process U^f by $U_t^f := f(t, S_{t-}, X_t)$. We want to compute ${}^{(p,\mathcal{F}^S)}U^f$. Consider for $0 \le t \le \overline{T}$ the conditional expectation $E[U_t^f | \mathcal{F}_t^S]$. It is well-known from the theory of nonlinear filtering that there is a right-continuous process Y which is a version of this conditional expectation, and which is continuous except at the jump-times T_n of the process N. By Theorem VI.7.10 in Rogers and Williams (1987) Y is indistinguishable from ${}^{(o,\mathcal{F}^S)}U^f$. Denote by $\left({}^{(o,\mathcal{F}^S)}U^f\right)^-$ the left-continuous version of this process. As the doubly-stochastic Poisson process N is quasi left-continuous, $\Delta N_\tau = 0$ for every predictable stopping time τ; hence

$${}^{(o,\mathcal{F}^S)}U_\tau^f = \left({}^{(o,\mathcal{F}^S)}U^f\right)_\tau^- \quad \text{for every predictable stopping time } \tau . \tag{3.14}$$

By definition of the optional prediction we have for any $\{\mathcal{F}_t^S\}$-stopping time τ the equality

$$E\left[U_\tau^f 1_{\{\tau<\infty\}}\right] = E\left[{}^{(o,\mathcal{F}^S)}U_\tau^f 1_{\{\tau<\infty\}}\right] .$$

Together with (3.14) this implies that for every predictable stopping time τ

$$E\left[U_\tau^f 1_{\{\tau<\infty\}}\right] = E\left[\left({}^{(o,\mathcal{F}^S)}U_\tau^f\right)^- 1_{\{\tau<\infty\}}\right] . \tag{3.15}$$

Being $\{\mathcal{F}_t^S\}$-adapted and left-continuous, the process $\left({}^{(o,\mathcal{F}^S)}U_\tau^f\right)^-$ is obviously $\{\mathcal{F}_t^S\}$-predictable. Together with (3.15) this implies that $\left({}^{(o,\mathcal{F}^S)}U_\tau^f\right)^-$ is indistinguishable from the predictable projection ${}^{(p,\mathcal{F}^S)}U^f$. Combining this result with (3.13) we get the following formula for our hedge-strategy

$$\xi_t^{(\mathcal{F}^S)} = \frac{\left(E[S_{t-}\lambda(t, X_t) g(t, S_{t-}, X_t) | \mathcal{F}_{t-}^S]\right)^-}{\left(E[\sigma^2 \S_{t-}^2 \lambda(t, X_t) | \mathcal{F}_{t-}^S]\right)^-} ,$$

which yields (3.11). □

Relation to nonlinear filtering: In order to compute our strategy we need to evaluate conditional expectations of the form

$$E[f(t, S_{t-}, X_t) | \mathcal{F}_t^S] \tag{3.16}$$

for continuous f. Now, the sequence $(Z_n)_n$ being independent of X, the conditional distribution of X_t given $\mathcal{F}_t^S$ equals the conditional distribution of X_t given past observations of

N, i.e. $\mathcal{F}_t^N$. Moreover, S_{t-} is observable for our agent at time t. Hence the computation of (3.16) reduces to computing

$$E[\tilde{f}(X_t)|\mathcal{F}_t^N] \tag{3.17}$$

for bounded and continuous functions $\tilde{f}$. The problem (3.17) has received some attention in the literature on nonlinear filtering. In particular, under Assumptions A1 and A2 there exist efficient recursive approximations to (3.17), which are derived using the reference-probability approach to nonlinear filtering. We refer the reader to Di Masi and Runggaldier (1982) and Frey and Runggaldier (1999a) for details.

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