Perfect Option Hedging for a Large Trader *

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Abstract

Standard derivative pricing theory is based on the assumption of agents acting as price takers on the market for the underlying asset. We relax this hypothesis and study if and how a large agent whose trades move prices can replicate the payoff of a derivative security. Our analysis extends prior work of Jarrow to economies with continuous security trading. We characterize the solution to the hedge problem in terms of a nonlinear partial differential equation and provide results on existence and uniqueness of this equation. Simulations are used to compare the hedging strategies in our model to standard Black-Scholes strategies.

Keywords: Option pricing, Black-Scholes model, Hedging, Large trader, Feedback Effects

JEL classification: G12, G13

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1 Introduction

Ever since derivative asset analysis started with the pathbreaking papers of Black and Scholes (1973) and Merton (1973), both academics and practioners were concerned about the strong assumptions this theory imposes on the market for the underlying assets. Most of the work on derivative pricing assumes that this market is complete, frictionless and that agents act as price takers. Of course this is a stylized picture of real security markets. Therefore recent research has studied the consequences of relaxing one or more of these hypothesises.

In this paper we study the replication of derivative securities from the viewpoint of a large trader whose trades influence asset prices. The framework of our analysis is a continuous-time version of the setup introduced by Jarrow (1992) and Jarrow (1994). The class of models considered in our paper is marked by the interaction of a large agent whose trades

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affect asset prices and a reference trader who is representative of many “small” price-taking agents. Our setup covers many of the models for asset price formation that have recently been developed to study the feedback effects of dynamic hedging on the price process of the underlying asset including Frey and Stremme (1994), Frey and Stremme (1995), Platen and Schweizer (1995) and Papanicolaou and Sircar (1996). The pricing and hedging of options from the viewpoint of the large trader is already studied by Jarrow (1994), who analyzes in detail a modified version of the binomial model of Cox, Ross, and Rubinstein (1979). He shows that the binomial option pricing model remains valid; however, in his framework the transition probabilities of the equivalent martingale measure which determines the price of derivatives depend on the trading strategy of the large agent.

Here we extend some of Jarrow's results and show that even with continuous security trading it is possible to find hedging strategies for the large trader which have the potential to replicate the payoff of certain non-path-dependent derivative securities including options. In the binomial model considered by Jarrow this question boils down to recursively solving a finite number of equations, but in our continuous-time setting it becomes rather involved. Nonetheless working in continuous time allows us to give a rather succinct characterization of the solution to the option replication problem for the large trader in terms of a nonlinear partial differential equation (PDE). We provide conditions for existence and uniqueness of solutions and analyze the shape of the hedging strategies. It turns out that the qualitative properties of hedging strategies for options are unchanged. However, simulations demonstrate that there may be considerable quantitative differences. The departure from the paradigm of price-taking agents has another important consequence: The hedge cost incurred by the large trader is no longer a linear function on the space of contingent claims. Our simulations demonstrate in particular that for a call option the hedge cost per share increases with the total amount of shares replicated by the large trader. We sketch some implications of this finding for the pricing of derivative securities.

We believe an analysis of option hedging by the large trader to be interesting for a number of reasons. To begin with, the analysis of Frey and Stremme (1995) and Platen and Schweizer (1995) has shown that the implementation of a dynamic hedging strategy designed to replicate some payoff which is a nonlinear function of the underlying asset affects the volatility of the underlying asset's price process. As a consequence it is impossible for a large agent to perfectly replicate the payoff of a derivative security with nonlinear payoff by means of a standard Black-Scholes hedging strategy; see for instance Frey and Stremme (1995). This immediately raises the question, if there are more general strategies that have the potential to replicate certain derivative securities with nonlinear payoff, even if the feedback effect of their implementation on the price process of the underlying security is taken into account. As explained by Platen and Schweizer (1995) such a strategy corresponds to a fixed point of the following transformation of volatility: They assign with every model for the volatility used by the large trader in computing his hedge strategy (the input volatility model) the actual market volatility model that results if the large trader implements a hedging strategy based on the input volatility model. Platen and Schweizer ask, whether a fixed point for this transformation exists. Our result on the existence of hedging strategies shows that the answer to this question is affirmative, if certain conditions on market liquidity and the nonlinearity of the terminal payoff are satisfied.

Moreover, an analysis of the additional assumptions needed to solve the hedging problem for the large trader and a comparison of the hedging strategies with their classical counterparts permits assessing the robustness of the traditional theory with respect to the assumption that agents are "small" relative to the market. Finally, as mentioned in Jarrow (1994), an option pricing theory which departs from the price-taking paradigm could possibly explain some of the anomalies observed on real options markets such as the stochasticity or the "smile pattern" of implied volatilities. We do not address these issues here, but our analysis is a necessary prerequisite for tackling them.

Most of the literature on option hedging in a model where the implementation of the hedging strategy affects prices focuses on the effects of dynamic trading strategies on the volatility of the underlying asset. Here we only mention the papers Grossman (1988), Brennan and Schwartz (1989), Gennotte and Leland (1990), Frey and Stremme (1995), Platen and Schweizer (1995) and Grossman and Zhou (1996). Apart from the previously mentioned work of Jarrow (1994) the issue of perfect option replication from the viewpoint of a large trader is dealt with by Schönbucher (1993), Papanicolaou and Sircar (1996) and Cvitanic and Ma (1996). The first two papers derive a fully nonlinear PDE for the hedge cost of a derivative security in a model that falls within the class of models considered in the present paper. Papanicolaou and Sircar (1996) use a linearized version of this PDE for an interesting simulation study. However, none of these papers deals with the issue of existence or uniqueness of solutions to the PDE.

Cvitanic and Ma (1996) postulate as starting point for their analysis a diffusion model for the price process of the underlying asset where drift and diffusion coefficient depend on the asset price, the value of the large trader's hedge portfolio and on the *level* of his position in the underlying asset. They use Forward-Backward SDE's to study the replication of non-path-dependent derivative securities by the large trader. The difference between their model and our setup is as follows: In the class of models considered in our paper the main factor influencing the diffusion coefficient of the asset price process is not the level of the large trader's stock position but its variability as measured by the *derivative* of the trading strategy with respect to the asset price.[1] As a consequence we have to deal with a quasilinear PDE for the hedging strategy of the large trader whereas the key step in Cvitanic and Ma (1996) is to solve a quasilinear PDE for the value of his hedge portfolio.

The remainder of this paper is organized as follows: In the next section we introduce the framework for our analysis. In Section 3 we characterize the solution of the replication problem by means of a nonlinear PDE. In Section 4 we carry out a detailed analysis of this PDE. The simulation results are presented in Section 5. Section 6 finally concludes.

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[1] This follows immediately from the explicit formulae for the market volatility in the presence of dynamic hedging obtained in Frey and Stremme (1995) and in Platen and Schweizer (1995).

2 The Model

Our analysis uses the the framework proposed by Jarrow (1992) and Jarrow (1994), but in contrast to these papers we consider an economy with *continuous* security trading in some intervall $[0, T]$. Our setup can be described as follows:

We postulate an economy with two traded assets, a riskless one (typically a bond or a money market account), called "the bond" and a risky one (typically a stock or a stock index) called "the stock". We take the bond as a numeraire, thereby making interest rates implicit to our model. Moreover, we assume the market for the bond to be perfectly elastic. This reflects the fact that money markets are far more liquid than those for the typical risky asset we have in mind here. The price process of the stock, accounted in units of the numeraire, is denoted by $X = (X_t)_{0 \le t \le T}$. The stock is in positive supply; for convenience we normalize the total amount of outstanding stock to be equal to one.

There are two types of traders in our economy; the first type is termed *large trader* or *program trader* the second type is called *reference trader*. The large trader could either be a single large agent or a group of traders acting in unison. As only the aggregate demand of this (group of) traders matters for the questions studied in the present paper we will not distinguish between these two possible interpretations. The name "large trader" for this agent is justified by Assumption (A.1) below where it is assumed that the trades of this agent influence the equilibrium stock price. In our analysis the large trader is confined to running dynamic hedging strategies for derivative securities; we study whether he is able to synthesize the payoff of certain derivative securities, even if the implementation of these strategies affects the equilibrium stock price. The reference trader creates the economic environment for our analysis of option hedging by the program trader. In our model this type of traders is representative of many small agents. We therefore assume that the reference trader acts as a price taker.

Many of the models for asset price formation that have been proposed in the recent financial literature are characterized by the interaction of different types of agents. Here we are particularly interested in the models proposed by Frey and Stremme (1994), Frey and Stremme (1995), Platen and Schweizer (1995) and Papanicolaou and Sircar (1996). These papers, which are inspired by the temporary equilibrium approach to the derivation of diffusion models for asset price fluctuations of Föllmer and Schweizer (1993), study the feedback effect of dynamic hedging on the price process of the underlying asset. As in our paper these authors consider models marked by the interaction of a representative reference trader and a program trader or portfolio insurer. The demand of the reference trader at time t is modelled as a function $D(t, F_t, x)$, where F_t is the value of some fundamental state variable process $F = (F_t)_{0 \le t \le T}$ and where x is the proposed price of the stock. In Frey and Stremme (1995) the state variable F represents the aggregated income of the reference trader, in Platen and Schweizer (1995) it is some unspecified liquidity demand, in other papers F has an interpretation as fundamental value of the firm. Now suppose that at time t the program trader holds a fraction α of the total supply of the stock. The equilibrium price X_t is then determined by the equation

$$D(t, F_t, X_t) + \alpha = 1\,. \tag{2.1}$$

In the above models it can be shown that (2.1) admits a unique solution, hence X_t can be expressed as a function of t, F_t and α, i.e. $X_t = \psi(t, F_t, \alpha)$. For instance we have in Platen and Schweizer (1995)

$$\psi(t, f, \alpha,) = f \cdot \exp(\lambda\alpha)\,, \tag{2.2}$$

for some constant $\lambda < 0$.[2] The model of Frey and Stremme (1995), which is also used by Papanicolaou and Sircar (1996), leads to the reaction function

$$\psi(t, f, \alpha) = D^* \cdot f/(1-\alpha)\,. \tag{2.3}$$

for some constant $D^* > 0$.

Following Jarrow (1994) we take this *reaction function* ψ as primitive for our analysis. We model the state variable as a stochastic process $F = (F_t)_{0\le t\le T}$ on some underlying filtered probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions. We assume that at time t agents have access to the information contained in $\mathcal{F}_t$. We make

Assumption (A.1)

(i) The fundamental F is a is a solution to the SDE $dF_t = \eta F_t dW_t$ for some one-dimensional Brownian motion W on $(\Omega, \mathcal{F}, P)$.[3] *We take the filtration $(\mathcal{F}_t)_{0\le t\le T}$ to be the augmented filtration generated by W.*

(ii) The discounted stock price at time t is given by $X_t = \psi(t, F_t, \alpha_t)$ where α_t denotes the stock position of the large trader. The reaction function ψ is a smooth function taking values in $\mathbb{R}_+$ with domain A given by $A := [0, T] \times \mathbb{R}_+ \times I$. Here $I \subset \mathbb{R}$ denotes some open interval that represents those values of the large trader's stock position for which there exists an equilibrium on the stock market. We postulate that

$$\begin{aligned} 0 &< \psi_f(t, f, \alpha) := \frac{\partial}{\partial f}\psi(t, f, \alpha) \text{ for all } (t, f, \alpha) \in A \\ 0 &\neq \psi_\alpha(t, f, \alpha) := \frac{\partial}{\partial \alpha}\psi(t, f, \alpha) \text{ for all } (t, f, \alpha) \in A, , t < T\,. \end{aligned}$$

Apart from the already mentioned models Assumption (A.1) is consistent with the model proposed by Hart (1977) and with the framework used in Schönbucher (1993). This is of interest, as the latter paper considers the hedging of options in finitely liquid markets. Note that the condition $\psi_\alpha \neq 0$ reflects the market power of the large trader since it implies that his trades actually affect prices. Jarrow (1994) assumes that ψ_α is strictly positive. This appears natural from an economic viewpoint, as a positive ψ_α corresponds to a demand function $D(t, f, x)$ in (2.1) which is *decreasing* in the proposed price x. Also empirical evidence on the price impact of large block transactions strongly supports the assumption that $\psi_\alpha > 0$; see Holthausen and Leftwich (1987). In fact, with the exception of Platen and Schweizer (1995) all the models mentioned above work with demand functions leading

[2] Jarrow (1994) also studies (2.2), but he considers the case $\lambda > 0$.

[3] The choice of the dynamics for F is somewhat arbitrary; in particular the assumption that F has zero drift is made purely for notational convenience. Assuming that the fundamental state variable process is a geometric Brownian motion will however facilitate a comparison of our results to those of the standard Black-Scholes option pricing theory.

to $\psi_\alpha > 0$. Platen and Schweizer raise the issue of perfect option replication by a large trader in their paper. To make our analysis compatible with their framework we therefore allow for negative ψ_α and explicitly mention whenever a result hinges on the (palatable) assumption that $\psi_\alpha > 0$.

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3 The Replication of Non-Path-Dependent Derivative Securities

3.1 Generalities

In this section we study whether the program trader whose trades affect security prices is able to synthesize the payoff of certain contingent claims by dynamic hedging. Our main result is a characterization of hedging strategies in terms of a terminal value problem involving a nonlinear partial differential equation. We restrict ourselves to contingent claims with single payoff at date T which is of the special form $c(X_T)$, i.e. we consider only non-path-dependent European derivative securities. We make the following assumptions on c.

Assumption (A.2) *The function $c : \mathbb{R}_+ \to \mathbb{R}$ belongs to the class $\mathcal{C}^3(\mathbb{R}_+)$. The range of its first derivative c' is bounded and contained in the intervall I (the third component of the range of the reaction function), and the second derivative c'' has compact support in $\mathbb{R}_+$.*

REMARKS: Often we will moreover assume that c is convex. This additional assumption holds for most payoffs which are replicated in practice.[4] Our differentiability assumptions on c are more problematic, since they exclude the payoff of ordinary options. However, "idealized options" where the kink has been smoothed are within the scope of our analysis. Moreover, we may interpret c as an idealized description of the aggregated payoff of an option portfolio containing a multitude of contracts with many different strikes; see for instance Frey and Stremme (1995). As shown in Theorem 4.2 below, the restriction on the range of c' ensures that the large trader's demand will always be contained in I.

Next we consider admissible trading strategies for the program trader.

Definition 3.1 *(i) A process $(\alpha, \beta) = (\alpha_t, \beta_t)_{0 \le t \le T}$ giving the program trader's position in stock and bond will be termed an admissible trading strategy if it is an adapted RCLL process and if α is moreover a semimartingale.*

(ii) Let (α, β) be an admissible trading strategy. The gains from trade of this strategy up to time t are given by $G_t := \int_0^t \alpha_s^- d\psi(s, F_s, \alpha_s)$, where α^- denotes the left continuous version of α.

(iii) The value process of this strategy is given by

$$V_t := \alpha_t \cdot \psi(t, F_t, \alpha_t) + \beta_t \,. \tag{3.4}$$

The strategy is called selffinancing if $V_t = V_0 + G_t$ for all $0 \le t \le T$.

[4] This is obvious for standard options and forwards, and it is even the defining characteristic of the so-called *portfolio insurance* strategies; see for instance Brennan and Schwartz (1989).

(iv) *An admissible strategy* (α, β) *with value process* V^c *replicates the claim with payoff* $c(X_T)$ *if the strategy is selffinancing and if moreover* $V_T^c = c(\psi(T, F_T, \alpha_T))$ *P-a.s.. For* $t \in [0, T)$ *we call* V_t^c *the hedge cost of the claim.*

Note the additional qualification of α being a semimartingale in (i) which is not needed in standard derivative asset pricing theory; together with the smoothness of ψ it implies that the stock price process X is itself a semimartingale so that the stochastic integral appearing in our definition of the gains from trade is well-defined.

The definitions in (ii) and (iii) parallel the usual ones. However, in our setting linear combinations of selffinancing strategies for the large trader are typically no longer selffinancing. This has potentially important consequences for the pricing of derivative assets. In the standard pricing theory for derivative securities such as Harrison and Pliska (1981) it is assumed that agents act as price takers both on the market for the underlying asset and on the market for the derivative security. Under this assumption the price of a derivative security that can be synthesized by a dynamic trading strategy must be equal to the hedge cost. The standard justification for this assertion, which hinges on the price taking assumption, goes as follows. Suppose for instance that the price of the derivative security exceeds the hedge cost. Agents could now sell the derivative security and follow the corresponding hedge strategy, thereby earning the difference between the price of the derivative security and the hedge cost as a riskless profit. As agents act as price takers on both markets, linear combinations of selffinancing strategies involving the underlying asset and the derivative security remain selffinancing. By performing the above arbitrage on a very large scale an agent could therefore earn unbounded riskless profits. Clearly this is at odds with any type of economic equilibrium.

The picture changes if we drop the assumption that our agent acts as a price taker on the market for the underlying asset. We will see in the simulations of section 5, that in our setup the hedge cost per share of a derivative security increases with the total number of shares replicated by the large trader. Hence a difference between the price of a derivative security and the corresponding hedge cost will typically lead to riskless profits for the non price taking agent which are strictly positive but bounded. In many economic models agents earn strictly positive riskless profits in a (short-run) equilibrium — just think of the standard static Cournot oligopoly model or of the competitive equilibrium model with firms having strictly increasing marginal cost curves. Hence in our framework we can no longer conclude that the price of a derivative security equals the hedge cost.[5] This is why we prefer to call the process V^c in Definition 3.1 (iv) hedge cost and not price. Therefore in our setup additional assumptions on the market structure are needed to arrive at a fully specified pricing theory for derivative securities.

In the previous version of this paper, Frey (1996), we sketched a pricing theory for derivatives based on the synchronous market condition introduced by Jarrow (1994). As there are many other economically interesting approaches to derivative pricing in the presence of a large trader we think that this issue deserves a paper in its own right. Here we therefore

[5] This argument works of course only if we assume that the reference trader, who is a price taker by assumption, is unable to replicate derivatives for some reason not explicitely modeled here such as lack of sophistication or lack of trading technology.

concentrate on discussing the replication of derivative securities by the large trader and the properties of the corresponding hedge cost. In any case an analysis of these questions is a prerequisite for a pricing theory for derivatives in a market with a large trader.

As every admissible trading strategy $\alpha = (\alpha_t)_{0\le t\le T}$ in the stock can be turned into a selffinancing strategy by choosing $\beta_t := V_0 + \int_0^t \alpha_s^- d\psi(s, F_s, \alpha_s) - \alpha_t \cdot \psi(t, F_t, \alpha_t)$, finding an admissible strategy that replicates the payoff $c(X_T)$ is equivalent to finding a representation

$$c(\psi(T, F_T, \alpha_T)) = V_0 + \int_0^T \alpha_s^- d\psi(s, F_s, \alpha_s) \tag{3.5}$$

for some admissible strategy α. As the integrator in this stochastic integral depends on the integrand α this is a non-standard problem and it is not a priori clear that a solution to this problem exists. In fact, to show that (3.5) has a solution for certain nonlinear payoffs is the major contribution of this paper.

In the following we will represent the payoff of the derivative security c in the form $c(x) = \rho \cdot h(x)$ where ρ is such that $\sup\{|h'(x)|, x > 0\} = 1$. In case that the program trader is replicating idealized call options or forward contracts on a fraction λ of the total supply of the stock this representation of c is obviously given by $\rho := \lambda$ and $h(x) := c(x)/\lambda$. Motivated by these important examples we will interpret ρ as the market weight of the program trader. Guided by the form of the hedge ratio in the classical Black-Scholes model we seek a solution of the option replication problem (3.5) having the form

$$\alpha_t = \rho \cdot \phi(t, F_t) \text{ for } 0 \le t \le T \tag{3.6}$$

for some function $\phi : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$.[6] We shall always assume that ϕ belongs to the class $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}_+)$. Clearly such a strategy is admissible.

Remarks: At a first glance it might seem more natural to put

$$\alpha_t = \rho \cdot \phi(t, F_t) \text{ for } 0 \le t < T \text{ and } \alpha_T = 0 \tag{3.7}$$

for some $\phi \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}_+)$. This choice of the hedging strategy corresponds to a situation where the large trader liquidates his stock position at the terminal date. If he follows this strategy, the stock price process might exhibit a jump at the terminal date given by $\Delta X_T = \psi(T, F_T, 0) - \psi(T, F_T, \rho \cdot \phi(T, F_T))$. As ΔX_T is measurable with respect to $\mathcal{F}_{T-}$, the asset price process does not admit an equivalent local martingale measure whenever $P[\Delta X_T > 0] > 0$; cf (Jacod and Shiryaev 1987, Lemma 1.2.27). It follows therefore from Delbaen and Schachermayer (1994) that the price process admits a so-called "Free Lunch with Vanishing Risk" (a particular form of an arbitrage opportunity) for the price-taking reference trader. Therefore we prefer to work with strategies of the form (3.6). In case that the program trader is replicating (idealized) ordinary options the strategy (3.6) can be interpreted as physical delivery of the underlying asset at the maturity date.

In order to avoid problems arising from possible jumps of the stock price caused by the liquidation of the program trader's portfolio, in most economic models dealing with the

[6] Once we have found a solution ϕ of the option hedging problem (3.5), we may represent it in the usual manner as a function of time and asset price by inverting the reaction function.

effects of portfolio insurance it is assumed that at the terminal date T the stock price is exogenously given and equal to the fundamental value F_T; see for instance Grossman and Zhou (1996). In these models the uncertainty about F_T is gradually removed for the reference trader as $t \to T$ such that he becomes more and more aggressive. This implies the convergence $\psi(t, f, \alpha) \to f$ as $t \to T$. If the trading strategy of the program trader is locally bounded and if this convergence is locally uniform we get $\Delta X_T = 0$ even if the program trader liquidates his position at the terminal date. Therefore the terminal value of the program trader's portfolio is the same in this scenario, no matter if he liquidates his portfolio in $t = T$ or not, and there is no need to distinguish between strategy (3.6) and strategy (3.7). We remark that the above discussion is not purely academic but points to a serious practical problem in relatively illiquid markets. We learned from market practioners, that in such a market traders who know that some other market participant has to dissolve a large hedge portfolio in the near future actually try to profit from this information by "frontrunning" the anticipated trades of the hedger.

3.2 A Characterization of Replicating Strategies for the Large Trader

If the program trader uses a strategy of the form (3.6), the asset price process is given by $X_t = X^\phi(t, F_t)$, where X^ϕ is shorthand for the composite function $\psi(t, f, \rho \cdot \phi(t, f))$. Defining h_0 by $V_0 = \rho \cdot h_0$ we immediately get that (3.5) is equivalent to

$$h(X^\phi(T, F_T)) = h_0 + \int_0^T \phi(s, F_s) dX^\phi(s, F_s)\,. \tag{3.8}$$

Since the stock price is now a function of t and F_t, the gains from trade in (3.8) can be computed by Itô's lemma yielding the following equivalent version of (3.8)

$$\begin{aligned} h(X^\phi(T, F_T)) &= h_0 + \int_0^T \phi(s, F_s) \cdot \frac{\partial}{\partial f} X^\phi(s, F_s) dF_s + \qquad (3.9)\\ &+ \int_0^T \phi(s, F_s) \cdot \left(\frac{\partial}{\partial t} X^\phi(s, F_s) + \frac{1}{2}\eta^2 F_s^2 \frac{\partial^2}{\partial f^2} X^\phi(s, F_s) \right) ds\,. \end{aligned}$$

This representation gives rise to the following

Proposition 3.2 *Suppose we are given a strategy function $\phi \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}_+)$. Then ϕ satisfies equation (3.9) if there exists a function $H : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ with the following properties:*

(i) H belongs to $\mathcal{C}([0, T] \times \mathbb{R}_+) \cap \mathcal{C}^{1,2}([0, T) \times \mathbb{R}_+)$.

(ii) H satisfies the terminal condition $H(T, f) = h(X^\phi(T, f)) \quad \forall f \in \mathbb{R}_+$.

(iii) We have for the derivatives

$$\frac{\partial}{\partial f} H(t, f) = \phi(t, f) \cdot \frac{\partial}{\partial f} X^\phi(t, f) \tag{3.10}$$

$$\frac{\partial}{\partial t} H(t, f) + \frac{1}{2}\eta^2 f^2 \frac{\partial^2}{\partial f^2} H(t, f) = \phi(t, f) \cdot \left(\frac{\partial}{\partial t} X^\phi(t, f) + \frac{1}{2}\eta^2 f^2 \frac{\partial^2}{\partial f^2} X^\phi(t, f) \right) \tag{3.11}$$

At time t the hedge cost of the claim $c(X_T)$ is then given by $\rho H(t, F_t)$.

REMARK: We will refer to $H(t, F_t)$ as *hedge cost per share* of the derivative security.

PROOF: To prove this proposition simply apply Itô's Lemma to the function H and note that (3.10) and (3.11) imply (3.9). □

Under some technical conditions also the converse of Proposition 3.2 holds. This is of interest, since it enables us to provide a complete *characterization* of solutions to the hedging problem (3.8) satisfying certain regularity conditions. Suppose we are given a solution $\phi(t,f) \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}_+)$ of (3.8). To construct a function H as in Proposition 3.2 we proceed as in the standard option pricing theory and compute the value process of the hedge portfolio as conditional expectation of the terminal payoff with respect to the equivalent martingale measure Q for X. Since the Markov property of the process F is preserved under the transition from P to Q, this conditional expectation is given by some function $H(t, F_t)$. If this function is sufficiently smooth it will satisfy the requirements of Proposition 3.2. We now give a formal proof. Assume that ρ and ϕ are such that

$$\frac{\partial}{\partial f} X^\phi(t,f) > 0 \quad \forall t \in [0,T], f \in \mathbb{R}_+ . \tag{3.12}$$

Then we may define a function $\mu(t,f) : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\mu(t,f) := \left(\frac{\partial}{\partial t} X^\phi(t,f) + \frac{1}{2}\eta^2 f^2 \frac{\partial^2}{\partial f^2} X^\phi(t,f) \right) \cdot \left(\frac{\partial}{\partial f} X^\phi(t,f) \right)^{-1}$$

and a process $Z = (Z_t)_{0 \le t \le T}$ by

$$Z_t := \exp\left(- \int_0^T \mu(s,F_s) \cdot (\eta F_s)^{-1} \, dW_s - \frac{1}{2} \int_0^T \left(\mu(s,F_s) \cdot (\eta F_s)^{-1} \right)^2 ds \right) . \tag{3.13}$$

Suppose that $E[Z_T] = 1$. Then Z is a martingale and we may define a new probability measure Q on $\mathcal{F}_T$ by setting $dQ/dP := Z_T$. It follows from Girsanov's theorem that under Q the process

$$\tilde{W}_t := W_t + \int_0^t \mu(s,F_s) \cdot (\eta F_s)^{-1} \, ds$$

is a Brownian motion. We note that F and X solve the following equations

$$dF_t = \eta F_t d\tilde{W}_t - \mu(t,F_t) dt \tag{3.14}$$

$$dX_t = \eta \cdot F_t \cdot \frac{\partial}{\partial f} X^\phi(t,F_t) d\tilde{W}_t , \tag{3.15}$$

in particular the stock price process is a local martingale under Q. Now we may state the converse to Proposition 3.2:

Proposition 3.3 *Suppose that $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}_+)$ is a solution to the replication problem (3.9) satisfying (3.12) and that the following conditions are satisfied:*

(i) The SDE (3.14) is well-posed and Z defined in (3.13) satisfies $E[Z_T] = 1$.

(ii) Both the asset price process X from (3.15) and the gains from trade $\int_0^t \phi(s,F_s) dX^\phi(s,F_s)$ are Q-martingales (and not only local martingales).

(iii) There is a solution $u \in \mathcal{C}([0,T] \times \mathbb{R}_+) \cap \mathcal{C}^{1,2}([0,T) \times \mathbb{R}_+)$ of the terminal value problem

$$\frac{\partial}{\partial t}u(t,f) + \frac{1}{2}\eta^2 f^2 \frac{\partial^2}{\partial f^2}u(t,f) - \mu(t,f)\frac{\partial}{\partial f}u(t,f) = 0\,, \quad u(T,f) = h(X^\phi(T,f))\,. \tag{3.16}$$

Then there exists a function $H \in \mathcal{C}([0,T]\times \mathbb{R}_+) \cap \mathcal{C}^{1,2}([0,T)\times\mathbb{R}_+)$ satisfying Proposition 3.2 (ii) and (iii).

Proof: Denote by $Q^{(t,f)}$ the law of the solution of the SDE (3.14) starting at time t with initial value equal to f. Since this SDE is well-posed we know that under $Q^{(t,f)}$ the coordinate process is a time-inhomogeneous Markov process, see for instance (Karatzas and Shreve 1988, Theorem 5.4.20). Defining $H(t,f)$ by

$$H(t,f) := E^{Q^{(t,f)}}[h(X^\phi(T,F_T))]\,,$$

we therefore get $H(t,F_t) = E^Q[h(X^\phi(T,F_T))\,|\mathcal{F}_t]$. It follows from (3.14) and the Feynman-Kac representation theorem (Karatzas and Shreve 1988, Theorem 5.7.6) that H coincides with u and hence fulfills point (i) and (ii) of Proposition 3.2. Since ϕ solves the replication problem we moreover have

$$\begin{aligned} h(X^\phi(T,F_T)) &= h_0 + \int_0^T \phi(s,F_s)dX^\phi(s,F_s) \\ &= E^Q[h(X^\phi(T,F_T)\,|\mathcal{F}_t] + \int_t^T \phi(s,F_s)dX^\phi(s,F_s) \qquad (3.17)\\ &= H(t,F_t) + \int_t^T \phi(s,F_s)dX^\phi(s,F_s)\,, \qquad (3.18) \end{aligned}$$

where (3.17) follows since the gains from trade are a martingale. On the other hand, since $h(X^\phi(T,f)) = H(T,f)$, Itô's Lemma yields

$$\begin{aligned} h(X^\phi(T,F_T)) &= H(t,F_t) + \int_t^T \frac{\partial}{\partial f}H(s,F_s)dF_s \qquad (3.19)\\ &+ \int_t^T \left(\frac{\partial}{\partial t}H(s,F_s) + \frac{\eta^2}{2}F_s^2\frac{\partial^2}{\partial f^2}H(s,F_s)\right)ds\,. \end{aligned}$$

By equating (3.18) and (3.19) we see that H satisfies also Proposition 3.2 (iii). □

We want to use Proposition 3.2 to construct a solution to the replication problem (3.9). If a function H with (3.10) and (3.11) exists we know that

$$\begin{aligned} \frac{\partial}{\partial t}H(t,f) &= \phi(t,f)\cdot\left(\frac{\partial}{\partial t}X^\phi(t,f) + \frac{1}{2}\eta^2 f^2\frac{\partial^2}{\partial f^2}X^\phi(t,f)\right) \qquad (3.20)\\ &- \frac{1}{2}\eta^2 f^2\frac{\partial}{\partial f}\left(\phi(t,f)\cdot\frac{\partial}{\partial f}X^\phi(t,f)\right)\,. \end{aligned}$$

Now if H exists it satisfies the identity $\frac{\partial}{\partial t}\frac{\partial}{\partial f}H(t,f) = \frac{\partial}{\partial f}\frac{\partial}{\partial t}H(t,f)$, yielding the following integrability condition for ϕ

$$\begin{aligned} \frac{\partial}{\partial t}\left(\phi(t,f)\cdot\frac{\partial}{\partial f}X^\phi(t,f)\right) &= \frac{\partial}{\partial f}\Big[\phi(t,f)\cdot\left(\frac{\partial}{\partial t}X^\phi(t,f) + \frac{1}{2}\eta^2 f^2\frac{\partial^2}{\partial f^2}X^\phi(t,f)\right) \qquad (3.21)\\ &- \frac{1}{2}\eta^2 f^2\frac{\partial}{\partial f}\left(\phi(t,f)\cdot\frac{\partial}{\partial f}X^\phi(t,f)\right)\Big]\,. \end{aligned}$$

Elementary but tedious computations given in Appendix A.1 now lead to the following

Lemma 3.4 *A function $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}_+)$ satisfies (3.21) if and only if it is a solution to the following PDE*

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t}\phi(t,f) + \frac{1}{2}\eta^2 f^2 \left(1 + 2\rho \frac{\psi_\alpha}{\psi_f} \cdot \frac{\partial}{\partial f}\phi(t,f)\right) \frac{\partial^2}{\partial f^2}\phi(t,f) \\
&+ \frac{\eta^2}{\psi_f}\frac{\partial}{\partial f}\phi(t,f) \cdot \Big[f \cdot \psi_f - \psi_t + \frac{f^2}{2}\psi_{ff} \\
&+ \rho \frac{\partial}{\partial f}\phi(t,f)\left(f^2 \psi_{\alpha f} + f\psi_\alpha\right) + \left(\rho \frac{\partial}{\partial f}\phi(t,f)\right)^2 \frac{f^2}{2}\psi_{\alpha\alpha}\Big] .
\end{aligned} \tag{3.22}
$$

Remark: The PDE (3.22) is *quasilinear* in the terminology of Friedman (1964) or Ladyzenskaja, Solonnikov, and Ural'ceva (1968), that is the coefficients depend not only on time and space variables but also on the solution and its first derivative. If there are no feedback effects, that is if $\psi_\alpha \equiv 0$, and if moreover $\psi(t,f,0) = f \quad \forall t, f$ the PDE (3.22) boils down to the usual linear PDE which is satisfied by the hedge ratio in the Black-Scholes model.

In the next theorem we show how to construct solutions of the replication problem (3.9) as solutions to a terminal value problem involving the PDE (3.22) and give a characterization of all solutions that possess certain smoothness properties.

Theorem 3.5 *Suppose that Assumptions (A.1) and (A.2) hold. Then a strategy function ϕ belonging to $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}_+)$ solves the hedging problem with feedback (3.9) if it solves the PDE (3.22) and if the terminal value $\phi(T,f)$ satisfies the equation*

$$\phi(T,f) = h'(X^\phi(T,f)) \quad \forall f > 0 . \tag{3.23}$$

Conversely, if $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}_+)$ solves the hedging problem and if moreover the assumptions from Proposition 3.3 are satisfied for the corresponding asset price process, ϕ is a solution to the PDE (3.22), and $\phi(T,f)$ satisfies (3.23).

Remark: Note that ϕ appears on both sides of (3.23) such that the terminal¡values are only implicitely defined. We will deal with this issue in Lemma 3.6 below.

Proof: To prove the first statement we want to construct a function H satisfying the requirements of Proposition 3.2. If ϕ solves the PDE (3.22), equations (3.10) and (3.20) define a vector field that satisfies the integrability conditions. Since the domain $(0,T) \times \mathbb{R}_+ \subset \mathbb{R}^2$ is convex there exists a function H — uniquely defined up to a constant — with (3.10), (3.20) and hence (3.11). Since the derivatives of ϕ are continuous functions on the set $[0,T] \times \mathbb{R}_+$ the derivative $\frac{\partial}{\partial t}H(t,f)$ given by (3.20) is bounded on every strip of the form $[T-\delta, T] \times K$ where K is a compact subset of $\mathbb{R}_+$. Hence we may extend H to a continuous function on $[0,T] \times \mathbb{R}_+$. Moreover, H can be defined in such a way that $H(T,f_0) = h(X^\phi(T,f_0))$ for some $f_0 \in \mathbb{R}_+$.

It remains to prove that $\frac{\partial}{\partial f}H(T,f) = h'(X^\phi(T,f))$, since then also requirement (*iii*) of Proposition 3.2 is fulfilled. Since the product $\phi(t,f) \cdot \frac{\partial}{\partial f}X^\phi(t,f)$ converges locally uniformly to $\phi(T,f) \cdot \frac{\partial}{\partial f}X^\phi(T,f)$ as $t \to T$ we get

$$\phi(T,f) \cdot \frac{\partial}{\partial f}X^\phi(T,f) = \lim_{t \to T} \phi(t,f) \cdot \frac{\partial}{\partial f}X^\phi(t,f) = \lim_{t \to T} \frac{\partial}{\partial f}H(t,f) = \frac{\partial}{\partial f}H(T,f) .$$

On the other hand we have $\frac{\partial}{\partial f} h(X^\phi(T,f)) = h'(X^\phi(T,f)) \cdot \frac{\partial}{\partial f} X^\phi(T,f)$ such that the terminal condition (3.23) yields the desired equality of the derivatives.

To prove the converse statement we first note that Proposition 3.3 implies the existence of a smooth function H with (3.10) and (3.20) such that ϕ satisfies the integrability conditions and solves hence the PDE (3.22). The terminal condition must hold since again $\frac{\partial}{\partial f} H(T,f)$ and $\frac{\partial}{\partial f} h(X^\phi(T,f))$ must be equal. □

Schönbucher (1993) and Papanicolaou and Sircar (1996) have derived a terminal value problem involving a fully nonlinear PDE for the hedge cost of a derivative security in a model of the form (2.1). As the analysis of fully nonlinear PDE's is more difficult than that of quasilinear PDE's they do not show existence and uniqueness of solutions to this terminal value problem. Under some regularity conditions one could differentiate their PDE with respect to the price of the underlying security to obtain a quasilinear PDE for the hedge ratio to which the methods of section 4 can be applied.

By (3.23) the terminal value $\phi(T,f)$, which reflects the special form of the replicated payoff, is given by the solution ϕ^* to the equation

$$h'(\psi(T,f,\rho\cdot\phi^*)) = \phi^* \,. \tag{3.24}$$

Lemma 3.6 *Suppose that Assumption (A.2) holds. If then for some* $\delta > 0$

$$\rho \cdot \sup\left\{h''(\psi(T,f,\rho\cdot\phi)) \cdot \psi_\alpha(T,f,\rho\cdot\phi),\ f \in \mathbb{R}_+, \phi \in I\right\} < 1-\delta \,, \tag{3.25}$$

there exists for every $f \in \mathbb{R}_+$ *a unique solution* $\phi^*(f)$ *to (3.24). The function* $f \mapsto \phi^*(f)$ *is twice continuously differentiable with bounded derivatives. If the terminal payoff* h *is convex its first derivative is positive.*

Proof: Existence of a solution follows since for all $f \in \mathbb{R}_+$ the mapping $\phi \mapsto \phi - h'(\psi(T,f,\rho\cdot\phi))$ is continuous. Condition (3.25) implies that this mapping is strictly increasing which implies uniqueness. Differentiability follows from the Implicit Function Theorem. By differentiating both sides of (3.24) we get

$$\frac{\partial}{\partial f}\phi^*(f) = \frac{h''\left(\psi(T,f,\rho\cdot\phi^*)\right)\cdot\psi_f(T,f,\rho\cdot\phi^*)}{1-\rho h''\left(\psi(T,f,\rho\cdot\phi^*)\right)\cdot\psi_\alpha(T,f,\rho\cdot\phi^*)}\,,$$

which is positive by (3.25) and the convexity of h. Boundedness of the derivatives of the function $\phi^*(f)$ follows, since h'' has compact support by (A.2). □

Remark: The compactness of the support of h'' implies that there is always some $\rho_0 > 0$ such that (3.24) holds for all $\rho \le \rho_0$. æ

4 Analysis of the PDE for the Hedging-Strategy

In this section we prove existence and uniqueness of a solution to the terminal value problem given by the PDE (3.22) and the terminal condition

$$\phi(T,f) = g(f) \ \text{ for } f > 0 \,. \tag{4.26}$$

We make the following regularity assumptions on the terminal values.

Assumption (A.3) *The function* $\tilde{g} : \mathbb{R} \to \mathbb{R}$*,* $x \mapsto g(\exp(x))$ *belongs to* $\mathcal{C}^2(\mathbb{R})$*, its derivatives are bounded and* $\tilde{g}''$ *is Hölder-continuous on* $\mathbb{R}$ *for some Hölder-exponent* $\beta \in (0,1)$*. Moreover* $\sup_{f\in\mathbb{R}_+} |g(f)| \leq 1$*.*

Note that by Lemma 3.6 (A.3) is satisfied if g is given by the function ϕ^* defined by the terminal condition (3.23). In that case $\frac{\partial}{\partial f} g(f) \geq 0$, provided that h is convex. Before we can state the main result of this section we have to specify the regularity conditions we impose on our solutions.

Definition 4.1 *Let* β *be some number from the interval* $(0,1)$*.*

A function $u \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ *is said to be Hölder-continuous of class* $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$*, if* u *and its derivatives are bounded on* $[0,T]\times\mathbb{R}$*, and if moreover the derivatives* u_x*,* u_{xx} *and* u_t *satisfy a Hölder condition in* x *with exponent* β *and a Hölder condition in* t *with exponent* $\beta/2$*. A function* $\phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}_+)$ *belongs to the space* $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R}_+)$ *if the function* u *defined by* $u(t,x) := \phi(t,\exp(x))$ *belongs to* $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$*.*

A more formal definition of this and related Hölder spaces is given in (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, chapter 1). To guarantee existence of a solution to the terminal value problem we have to impose the following restrictions on the reaction function ψ.

Assumption (A.4) *For every compact set* $K \subset I$ *there are finite constants* $K_1,\ldots,K_5$ *such that for all* $t\in[0,T], f>0, \alpha\in K$

$$
\begin{aligned}
&|\psi_\alpha(t,f,\alpha)/(f\cdot\psi_f(t,f,\alpha))| < K_1, && |\psi_{\alpha\alpha}(t,f,\alpha)/f\cdot\psi_f(t,f,\alpha))| < K_2,\\
&|\psi_{f\alpha}(t,f,\alpha)/\psi_f(t,f,\alpha)| < K_3, && |f\cdot\psi_{ff}(t,f,\alpha)/\psi_f(t,f,\alpha)| < K_4,\\
&|\psi_t(t,f,\alpha)/(f\cdot\psi_f(t,f,\alpha))| < K_5\,. &&
\end{aligned}
$$

REMARKS: The constants $K_1,\ldots,K_3$ can be interpreted as measures of market liquidity. Assumption (A.4) is always satisfied if — as in the models of Jarrow (1994), Platen and Schweizer (1995) or Frey and Stremme (1995) — the reaction function is of the particular form $\psi(t,f,\alpha) = \widetilde{\psi}(t,\alpha)\cdot f^\lambda$ for some $\lambda > 0$. It holds true for other reaction functions with similar asymptotic properties for $f\to 0$ and $f\to\infty$, too.

Theorem 4.2 *Suppose that Assumptions (A.1), (A.4) hold for the reaction function* ψ*, that the terminal values satisfy (A.3) for some Hölder exponent* β *and that moreover* $\frac{\partial}{\partial f} g(f) \geq 0$*. Then the following holds*

(i) There is some $0 < \bar\rho \leq 1$ *such that for every* $\rho \leq \bar\rho$ *the terminal value problem (3.22), (4.26) has a solution contained in* $H^{2+\beta,1+\beta/2}$ $([0,T]\times\mathbb{R}_+)$*.*

(ii) Every solution $\phi \in H^{2+\beta,1+\beta/2}$ $([0,T]\times\mathbb{R}_+)$ *of this terminal value problem that satisfies the following condition*

$$
\exists \delta > 0 \text{ with } 2\rho\cdot\frac{\partial}{\partial f}\phi(t,f)\cdot\frac{\psi_\alpha(t,f,\rho\cdot\phi(t,f))}{\psi_f(t,f,\rho\cdot\phi(t,f))} \geq \delta - 1 \quad t\in[0,T), f>0 \tag{4.27}
$$

has the following properties:

$$
\inf_{f\in\mathbb{R}_+} g(f) \leq \phi(t,f) \leq \sup_{f\in\mathbb{R}_+} g(f) \;\text{ and }\; \frac{\partial}{\partial f}\phi(t,f) > 0\,. \tag{4.28}
$$

If $\psi_\alpha \geq 0$ (4.28) holds for all solutions $\phi \in H^{2+\beta,1+\beta/2}$ $([0,T] \times \mathbb{R}_+)$; condition (4.27) is not needed.

(iii) For every $\rho \in [0,1)$ there is at most one solution of the terminal value problem (3.22), (4.26) that belongs to $H^{2+\beta,1+\beta/2}$ $([0,T] \times \mathbb{R}_+)$ and satisfies (4.28). If $\psi_\alpha \geq 0$ this terminal value problem has a unique solution in the class $H^{2+\beta,1+\beta/2}$ $([0,T] \times \mathbb{R}_+)$.

REMARKS: To guarantee existence of a solution to the terminal problem for the hedge ratio we have to restrict the market weight ρ of the large trader. This additional qualification is needed, since when dealing with nonlinear PDE's one has to impose certain restrictions on the character of the nonlinear occurrences of the solution and its first derivative in the coefficients of the equation, cf. (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, chapter 1.3). The constant $\bar{\rho}$ depends essentially on two factors, first on the "Gamma" of the terminal payoff, that is on $\sup\{g'(f) \cdot f,\ f \in \mathbb{R}_+\}$ and second on the liquidity of the market as measured by the K_i in (A.4). Statement (*ii*) implies that the qualitative properties of the hedge ratio are unaltered by working in finitely elastic markets. However, we will see in the simulations of section 5 that there are quantitative differences which may be quite large. In case that $\psi_\alpha < 0$ condition (4.27) is needed to ensure that we can transform the PDE (3.22) into a parabolic PDE on $[0,T] \times \mathbb{R}$. Note that condition (4.27) implies in particular that $\frac{\partial}{\partial f} X^\phi(t,f) > 0\ \forall (t,f) \in [0,T] \times \mathbb{R}^+$.

The rest of this section is devoted to the proof of Theorem 4.2. Our main tools are results from Ladyzenskaja, Solonnikov, and Ural'ceva (1968). To apply these results we have to transform the terminal value problem on $\mathbb{R}_+$ into an initial value problem on $\mathbb{R}$. To this end we introduce the new time variable $\tau(t) = T - t$ and the new space variable $x(f) = \ln(f)$. We define a function $u : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ by $\phi(t,f) =: u(\tau(t), x(f))$. Elementary calculations show that ϕ solves the terminal value problem (3.22), (4.26) if and only if the function u is a solution of the Cauchy problem

$$u_t = a(\rho,t,x,u,u_x) \cdot u_{xx} + b(\rho,t,x,u,u_x) \cdot u_x \tag{4.29}$$
$$u(0,x) = \tilde{g}(x)\,, \tag{4.30}$$

where the functions a and b are given by

$$a(\rho,t,x,u,q) = \frac{1}{2}\eta^2 \cdot \left(1 + 2 \cdot \rho \cdot q \cdot e^{-x} \cdot \frac{\psi_\alpha(T-t,e^x,\rho \cdot u)}{\psi_f(T-t,e^x,\rho \cdot u)}\right) \tag{4.31}$$
$$b(\rho,t,x,u,q) = \frac{\eta^2}{\psi_f} \cdot \left(\frac{1}{2}\psi_f + \frac{1}{2} \cdot e^x \psi_{ff} - e^{-x}\psi_t + \rho q \psi_{f\alpha} + \frac{\rho^2 q^2}{2} e^{-x} \cdot \psi_{\alpha\alpha}\right) \tag{4.32}$$

In (4.32) the arguments of ψ and its derivatives are given by $(T-t, e^x, \rho \cdot u)$, too. From now on we concentrate on the initial value problem (4.29) (4.30). For technical reasons we have to introduce truncated versions $\bar{a}$ and $\bar{b}$ of our coefficients. Their precise definition is given in the Appendix A.2. For an appropriate choice of $\bar{a}$ the PDE

$$u_t = \bar{a}(\rho,t,x,u,u_x) \cdot u_{xx} + \bar{b}(\rho,t,x,u,u_x) \cdot u_x \tag{4.33}$$

is *parabolic*, so that we can apply results from the theory of quasilinear parabolic PDE's to our problem. In the next proposition we establish existence and uniqueness of the Cauchy problem (4.33) (4.30). After that we will prove certain properties of the solutions, thereby showing that for ρ sufficiently small a solution to the PDE (4.33) solves also the original equation (4.29), which proves Theorem 4.2.

Proposition 4.3 *Suppose that the initial values g satisfy Assumption (A.3).*

(i) Then for all $\rho \in [0, 1)$ *there is at least one solution u to the Cauchy problem (4.33), (4.30) belonging to* $H^{2+\beta,1+\beta/2}([0, T] \times \mathbb{R})$*.*

(ii) For every $0 \le \rho_0 < 1$ *there is some constant K depending only on the Hölder norm of the initial values* $\tilde{g}$ *and on the size of the constants in Assumption (A.4), such that* $|u_x(t, x)| < K$ *for all* $0 \le \rho \le \rho_0$, $t \in [0, T]$, $x \in \mathbb{R}$*.*

(iii) For any $\beta \in (0, 1)$ *and any* $\rho \in [0, 1]$ *there is at most one solution of the Cauchy problem (4.33), (4.30) belonging to* $H^{2+\beta,1+\beta/2}([0, T] \times \mathbb{R})$*.*

The proof consists of an application of (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.8.1) and is given in Appendix A.2.

In the following proposition we prove the most important properties of solutions to the Cauchy problem (4.33), (4.30) and show that they carry over to the unrestricted Cauchy problem (4.29), (4.30).

Proposition 4.4 *Every solution of the Cauchy problem (4.33), (4.30) belonging to some Hölder space* $H^{2+\beta,1+\beta/2}([0, T] \times \mathbb{R})$ *for some* $\beta > 0$ *satisfies* $\forall\, (t, x) \in [0, T] \times \mathbb{R}$:

$$\inf_{x\in\mathbb{R}} \tilde{g}(x) \le u(t, x) \le \sup_{x\in\mathbb{R}} \tilde{g}(x) \text{ and } u_x(t, x) \ge 0\,.$$

If either $\psi_\alpha \ge 0$ *or if condition (4.27) is satisfied these properties carry over to solutions of the unrestricted Cauchy problem (4.29), (4.30) belonging to* $H^{2+\beta,1+\beta/2}([0, T] \times \mathbb{R})$ *for some* $\beta > 0$*.*

Proof: The key of the proof is the following observation: Whenever u solves the quasilinear PDE (4.33), it is also a solution to the following *linear* parabolic equation

$$u_t = a^u(\rho, t, x)u_{xx} + b^u(\rho, t, x)u_x\,, \tag{4.34}$$

where $a^u(\rho, t, x) := \bar{a}(\rho, t, x, u(t, x), u_x(t, x))$ and $b^u(\rho, t, x) := \bar{b}(\rho, t, x, u(t, x), u_x(t, x))$. The bounds on u follow therefore directly from the maximum principle for linear parabolic PDE's, or they can be read from the Feynman-Kac representation of u.

To prove the positivity of u_x we first note that (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 3.12.2) can be applied to the PDE (4.34), yielding $u \in H^{3+\beta,(3+\beta)/2}((0, t) \times \Omega)$ for every $\Omega \subset\subset \mathbb{R}$. In particular the derivatives $\frac{\partial}{\partial t}\frac{\partial}{\partial x}u$ and $\partial^3 u/\partial x^3$ are well-defined. Hence by differentiating (4.34) we obtain the following linear parabolic PDE for $v(t, x) := u_x(t, x)$

$$v_t = a^u(\rho, t, x)v_{xx} + v_x \cdot \left(\frac{\partial}{\partial x}a^u(\rho, t, x) + b^u(\rho, t, x)\right) + v \cdot \frac{\partial}{\partial x}b^u(\rho, t, x)\,.$$

Of course v is continuous on $[0, T] \times \mathbb{R}$ and has initial values $v(0, x) := \frac{\partial}{\partial x}g(\exp(x))$ which are nonnegative by assumption. It can be checked that the regularity conditions for the

Feynman-Kac theorem are fulfilled. Therefore we obtain the following stochastic representation for v:

$$v(t,x) = E^{(T-t,x)}\left[v(0,Y_T)\cdot \exp\left(\int_{T-t}^{T}\frac{\partial}{\partial x}b^u(s,Y_s)ds\right)\right] \tag{4.35}$$

where Y solves the SDE

$$dY_t = \sqrt{a^u(\rho,t,Y_t)}dW_t + \left(b^u(\rho,t,Y_t) + \frac{\partial}{\partial x}a^u(\rho,t,Y_t)\right)dt\,.$$

Since all the terms in (4.35) are nonnegative it follows immediately that $v(t,x) \geq 0$.

Now let us turn to the second claim. Consider first the case $\psi_\alpha > 0$. Suppose we are given a function $u \in H^{2+\beta,1+\beta/2}([0,T]\times \mathbb{R})$ solving the PDE (4.29) with initial values (4.30), for which there exists some (t_0,x_0) with $u_x(t_0,x_0) < 0$. Since $u_x(0,x) \geq 0$, and since by definition of the Hölder space $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$ u_x is Hölder continuous in t uniformly in x, there is then for every $\varepsilon > 0$ some pair (t^*,x^*) with

- $\forall\, t \leq t^*,\ \forall\, x \in \mathbb{R} \quad u_x(t,x) > -\varepsilon/2.$
- $u_x(t^*,x^*) < 0\,.$

On the other hand for appropriate truncation functions u is also a solution to the restricted Cauchy problem (4.33), (4.30) on $[0,t^*]\times\mathbb{R}$. Hence $u_x(t^*,x^*) \geq 0$, a contradiction. As $u_x \geq 0$ u is also a solution of the PDE (4.29) if we choose appropriate truncation functions, such that the bounds on the range of u follow from the properties of solutions to (4.33), (4.30).

Next turn to the case $\psi_\alpha < 0$. By (4.27) we can find for a function $u \in H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$ solving the PDE (4.29) truncation functions so that u solves also the PDE (4.33). Hence the claim follows again from the properties of solutions to (4.33), (4.30) established in the first part of the proof. □

Remark: The proof of the positivity of u_x shows that option prices are convex functions of the price of the underlying security in any Markovian model where the stock price follows a diffusion equation of the form $dX_t = \sigma(t,X_t)X_t dW_t$ for a sufficiently smooth function σ. This result has independently been proven by Bergman, Grundy, and Wiener (1995) and, using probabilistic arguments, by El Karoui, Jeanblanc-Picqué, and Shreve (1995) and Hobson (1996).

It is now easy to proof Theorem 4.2 using Propositions 4.3 and 4.4; see Appendix A.2.

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5 Results from Simulations

In case there are no feedback effects from the large trader's position into equilibrium prices our theory for option pricing and hedging boils down to the standard theory as developed for instance by Black and Scholes (1973) and Harrison and Pliska (1981). Using explicit numerical computations we now want to compare in the case of a call option the hedging strategy and the hedge cost per share obtained in our model with a large trader to option prices and hedging strategies in the Black-Scholes model. In all simulations we work with the reaction function $\psi(t, f, \alpha) = f/(1-\alpha)$ introduced in section 2 and with a terminal payoff given by

$$h(x) = \frac{1}{2}\left(x - K + \sqrt{(x-K)^2 + \alpha}\right)$$

for some small $\alpha > 0$, i.e. we are considering call options with "smoothed kink". For these data there exists a solution to the option replication problem (3.8) by Theorems 3.5 and 4.2. To numerically solve the PDE (3.22) we used the method of finite differences as explained for instance in (Wilmott, Dewynne, and Howison 1993, Chapter 19). By Theorem 4.2 (*ii*) we know that $\frac{\partial}{\partial f}\phi(t, f) > 0$. Therefore we get $\frac{\partial}{\partial f}X^{\phi}(t, f) > 0$. Hence for every fixed t the fundamental f can be expressed as $(X^{\phi})^{-1}(t, x)$. This allows us to represent also the hedging strategy as a function $\varphi(t, x)$ of the equilibrium price:

$$\phi(t, f) = \phi\left(t, (X^{\phi})^{-1}(t, X^{\phi}(t, f))\right) =: \varphi(t, X^{\phi}(t, f)). \tag{5.36}$$

The first simulation we have run illustrates that the qualitative properties of the hedging strategy remain unaltered, a fact we have proven already in point (*ii*) of Theorem 4.2. In figure 1 we have graphed the solution of our hedge problem as function $\varphi(t, x)$ of time t and price x of the underlying asset for $\rho = 0.2$. The plot looks very similar to the usual pictures of the "Delta" in the Black-Scholes model. However, there are quantitative differences, as it is shown by figure 2 where we have plotted the hedge ratio at $t = 1$ as a function of x for different values of ρ.

We have also computed H_0, the hedge costs per share as defined in Proposition 3.2, for different values of ρ.[7] The results of the simulations are plotted in figure 3 and figure 4. It is obvious that the hedge cost per share increases with increasing ρ. Comparing figure 3 where we have plotted H_0 against the fundamental f to figure 4 where we have varied the underlying's price x we see that in figure 3 the increase in the hedge cost per share caused by a rise in ρ is much more pronounced. This comparison reveals two reasons for the increase in the hedge cost. First an increasing ρ implies an increase in the large trader's stock position and hence increasing asset prices. Second the rise in ρ causes a rise in stock price volatility, which explains why the hedge cost increases even if the asset price is kept constant. This finding is in sharp contrast to the standard option pricing theory with price-taking agents where the hedge cost per share of the derivative security is independent of the overall amount of hedging; its implications for the pricing of derivative securities are discussed in section 3.1.

6 Conclusion

In this paper we have studied if and how a large trader whose trades affect asset prices can synthesize the payoff of certain derivative securities by a dynamic hedging strategy. We have characterized this hedging strategy as solution of a nonlinear PDE which generalizes the usual PDE satisfied by the hedging strategy in standard option pricing theory. Additional

[7] To compute H we numerically integrated equation (3.10) and used $H(t, 0) \equiv 0$.

qualifications not needed in the standard theory are necessary to guarantee existence of a solution to this PDE. It turned out that the qualitative shape of the hedge ratio is unaltered by dropping the price taking assumption, but simulations revealed that there are qualitative differences which may be quite large. Moreover, we found that for a call option the hedge cost per share increases with the overall number of shares replicated by the large trader. We discussed some implications of this finding for the pricing of derivative securities in a market with non-price-taking agents. This is an interesting area for further research.

7 Results from Simulations

$\varphi(t,x)$

$\varphi(t,x)$

1

0.8

0.6

0.4

0.2

0

0

t 0.5

1

0.5 1 1.5 x 2 2.5 3

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Figure 1: Hedge ratio as function of price and time for a value $\rho = 0.2$

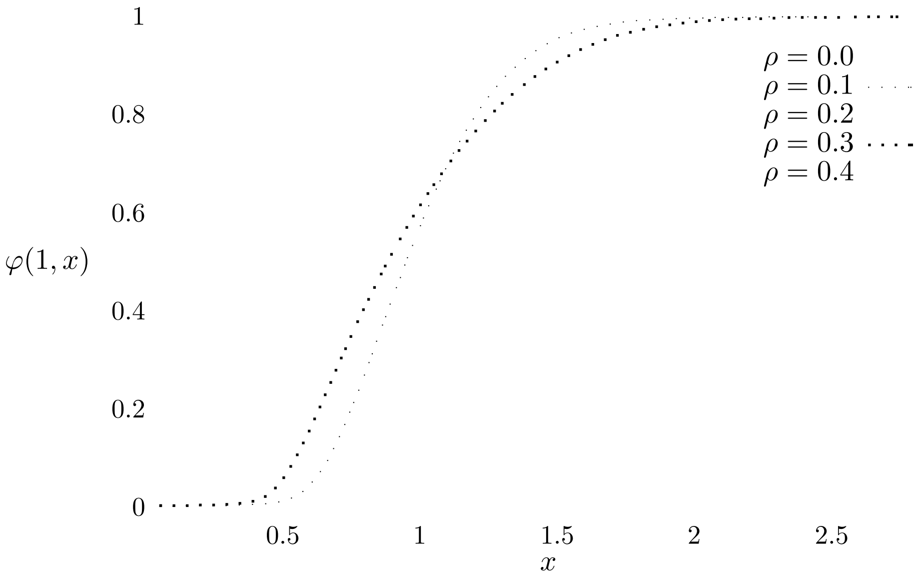

Figure 2: Hedge ratio $\varphi(1,x)$ at $t = 1$ for different values of ρ

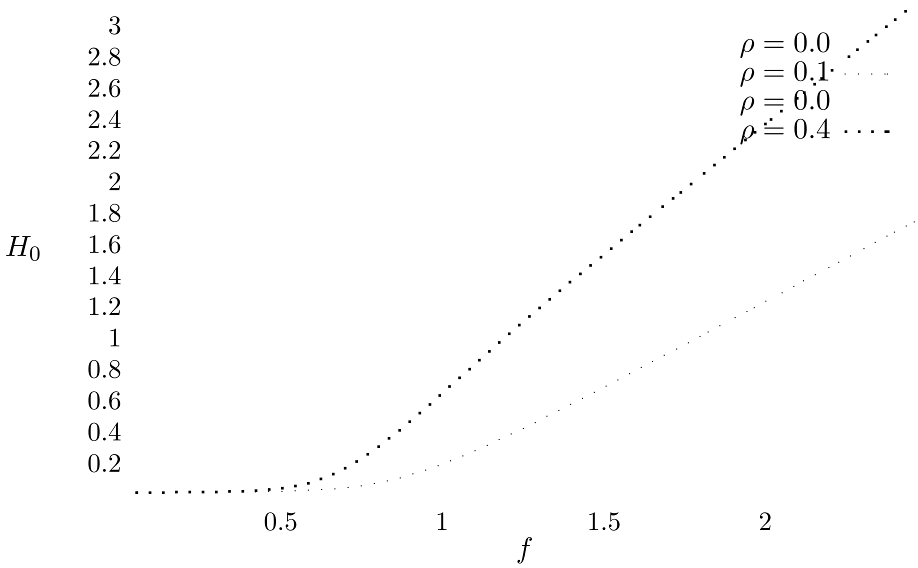

Figure 3: Cost of hedging per contract H_0 as a function of the fundamental f for different values of ρ

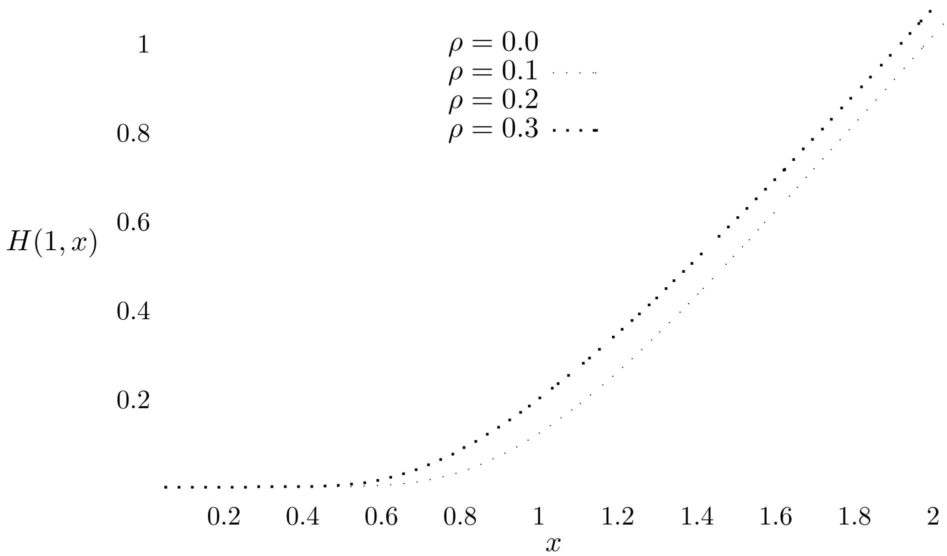

Figure 4: Cost of hedging per contract H_0 as a function of the current price x for different values of ρ

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A Mathematical Appendix

A.1 Proof of Lemma 3.4

The following computations show that (3.21) implies (3.22), to prove the converse direction one has to go through the computations in reverse order. To shorten the notation we will always omit the arguments (t, f). We start by computing $\frac{\partial}{\partial t}(\phi \cdot \frac{\partial}{\partial f} X^\phi)$ and get

$$\frac{\partial}{\partial t}(\phi \cdot \frac{\partial}{\partial f} X^\phi) = \frac{\partial}{\partial t}\phi \cdot \frac{\partial}{\partial f} X^\phi + \phi \cdot \frac{\partial^2}{\partial f \partial t} X^\phi \tag{A.37}$$

Now we turn to calculating the right hand side of (3.21). We get

$$\begin{aligned}\phi\left(\frac{\partial}{\partial t}X^\phi + \frac{1}{2}\eta^2 f^2 \frac{\partial^2}{\partial f^2}X^\phi\right) \quad &- \quad \frac{1}{2}\eta^2 f^2 \frac{\partial}{\partial f}\left(\phi \cdot \frac{\partial}{\partial f}X^\phi\right) = \\ &= \quad \phi \cdot \frac{\partial}{\partial t}X^\phi - \frac{1}{2}\eta^2 f^2 \cdot \frac{\partial}{\partial f}\phi \cdot \frac{\partial}{\partial f}X^\phi \end{aligned} \tag{A.38}$$

Computation of the derivative of (A.38) wrt. f now yields the right hand side of (3.21). Here we get

$$\begin{aligned}\phi \cdot \frac{\partial^2}{\partial f \partial t}X^\phi \quad &+ \quad \frac{\partial}{\partial f}\phi \cdot \frac{\partial}{\partial t}X^\phi - \eta^2 f \cdot \frac{\partial}{\partial f}\phi \cdot \frac{\partial}{\partial f}X^\phi - \\ &-\frac{1}{2}\eta^2 f^2 \cdot \frac{\partial^2}{\partial f^2}\phi \cdot \frac{\partial}{\partial f}X^\phi - \frac{1}{2}\eta^2 f^2 \cdot \frac{\partial}{\partial f}\phi \cdot \frac{\partial^2}{\partial f^2}X^\phi\end{aligned} \tag{A.39}$$

Equating both sides of (3.21), i.e. (A.37) and (A.39) now yields the following PDE for ϕ:

$$\begin{aligned}\frac{\partial}{\partial t}\phi \cdot \frac{\partial}{\partial f}X^\phi + \phi \cdot \frac{\partial^2}{\partial f \partial t}X^\phi \quad &= \quad \phi \cdot \frac{\partial^2}{\partial f \partial t}X^\phi + \frac{\partial}{\partial f}\phi \cdot \frac{\partial}{\partial t}X^\phi - \eta^2 f \cdot \frac{\partial}{\partial f}\phi \cdot \frac{\partial}{\partial f}X^\phi \\ &- \quad \frac{1}{2}\eta^2 f^2 \cdot \frac{\partial^2}{\partial f^2}\phi \frac{\partial}{\partial f}X^\phi - \frac{1}{2}\eta^2 f^2 \cdot \frac{\partial}{\partial f}\phi \cdot \frac{\partial^2}{\partial f^2}X^\phi.\end{aligned} \tag{A.40}$$

Since

$$\begin{aligned}\frac{\partial}{\partial t}\phi \cdot \frac{\partial}{\partial f}X^\phi \quad &= \quad \frac{\partial}{\partial t}\phi \cdot \left(\psi_f + \rho \cdot \psi_\alpha \cdot \frac{\partial}{\partial f}\phi\right) \text{ and} \\ \frac{\partial}{\partial f}\phi \cdot \frac{\partial}{\partial t}X^\phi \quad &= \quad \frac{\partial}{\partial f}\phi \cdot \left(\psi_t + \rho \cdot \psi_\alpha \cdot \frac{\partial}{\partial t}\phi\right),\end{aligned}$$

cancelling terms on both sides yields the following version of the PDE (A.40).

$$\begin{aligned}\frac{\partial}{\partial t}\phi \cdot \psi_f \quad &= \quad \frac{\partial}{\partial f}\phi \cdot \psi_t - \eta^2 \cdot f \cdot \frac{\partial}{\partial f}\phi \cdot \frac{\partial}{\partial f}X^\phi \\ &-\frac{1}{2}\eta^2 f^2 \cdot \frac{\partial^2}{\partial f^2}\phi \cdot \frac{\partial}{\partial f}X^\phi - \frac{1}{2}\eta^2 f^2 \cdot \frac{\partial}{\partial f}\phi \cdot \frac{\partial^2}{\partial f^2}X^\phi\end{aligned}$$

Now we use that

$$
\begin{aligned}
\frac{\partial}{\partial f}X^{\phi} &= \psi_f + \rho\cdot\psi_\alpha\cdot\frac{\partial}{\partial f}\phi \\
\frac{\partial^2}{\partial f^2}X^{\phi} &= \psi_{ff} + \psi_{\alpha f}\cdot\rho\cdot\frac{\partial}{\partial f}\phi + \rho\cdot\frac{\partial}{\partial f}\phi\cdot\left(\psi_{\alpha f} + \psi_{\alpha\alpha}\cdot\rho\cdot\frac{\partial}{\partial f}\phi\right) + \rho\cdot\psi_\alpha\cdot\frac{\partial^2}{\partial f^2}\phi \\
&= \psi_{ff} + 2\psi_{\alpha f}\cdot\rho\cdot\frac{\partial}{\partial f}\phi + \rho^2\psi_{\alpha\alpha}\left(\frac{\partial}{\partial f}\phi\right)^2 + \rho\psi_\alpha\frac{\partial^2}{\partial f^2}\phi
\end{aligned}
$$

and obtain the following version of (A.40)

$$
\begin{aligned}
\psi_f\cdot\frac{\partial}{\partial t}\phi &= -\frac{1}{2}\eta^2 f^2\cdot\frac{\partial^2}{\partial f^2}\phi\cdot\left(\psi_f + \rho\cdot\psi_\alpha\cdot\frac{\partial}{\partial f}\phi + \rho\cdot\psi_\alpha\frac{\partial}{\partial f}\phi\right) \\
&- \frac{\partial}{\partial f}\phi\cdot\left(-\psi_t + \eta^2\cdot f\cdot\psi_f + \frac{1}{2}\eta^2 f^2\psi_{ff}\right) \\
&- \left(\frac{\partial}{\partial f}\phi\right)^2\cdot(\eta^2\cdot f^2\cdot\rho\cdot\psi_{\alpha f} + \eta^2\cdot f\cdot\rho\cdot\psi_\alpha) - \left(\frac{\partial}{\partial f}\phi\right)^3\cdot\left(\frac{1}{2}\eta^2\cdot\rho^2\cdot\psi_{\alpha\alpha}\right)
\end{aligned}
$$

Rearranging terms we see that this is the PDE from Lemma 3.4. □

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A.2 Complements to the Proof of Theorem 4.2

Definition of the truncated coefficients in the PDE (4.33)

We define the truncated versions $\bar{a}$ and $\bar{b}$ of the coefficient functions defined in (4.31) and (4.32) as follows. We set

$$
\begin{aligned}
\bar{a}(\rho,t,x,u,q) &:= \frac{1}{2}\eta^2\cdot\left(1 + c_1\left(2\cdot\rho\cdot q\cdot e^{-x}\cdot\frac{\psi_\alpha(T-t,e^x,\rho\cdot u)}{\psi_f(T-t,e^x,\rho\cdot u)}\right)\right) \\
\bar{b}(\rho,t,x,u,q) &:= b(\rho,t,x,u,\frac{1}{\rho}\cdot c_2(\rho q))
\end{aligned}
$$

Here $c_1,\ c_2:\ \mathbb{R}\to\mathbb{R}$ are smooth functions with

$$
\begin{array}{lll}
1\geq c_1'\geq 0, & c_1(y)=y \text{ on } [1-\varepsilon, M-\varepsilon/2], & c_1(y)\in[1-\varepsilon/2, M] \\
1\geq c_2'\geq 0, & c_2(y)=y \text{ on } [-M+\varepsilon/2, M-\varepsilon/2], & c_2(y)\in[-M,M],
\end{array}
\tag{A.41}
$$

where $\varepsilon > 0$ is small and M is some large positive constant. As the solutions to our Cauchy problems are contained in $I^g := [\inf_{f>0} g(f)\,,\,\sup_{f>0} g(f)]$ (see Proposition 4.4), by Assumption (A.4) we are able to find for every $0\leq\rho_0<1$ some ε in (A.41) which is small enough to ensure that there is some constant $\tilde{K}_0>0$ such that

$$
\inf\{\bar{a}(\rho,t,x,u,q),\ \rho\in[0,\rho_0], t\in(0,T], x\in\mathbb{R}, u\in I^g, q\in\mathbb{R}\} > \tilde{K}_0 \tag{A.42}
$$

Hence for this choice of c_1 the PDE (4.33) is parabolic.

Proof of Proposition 4.3

To prove the proposition we have to show that (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.8.1) can be applied to the Cauchy problem (4.33), (4.30). This theorem is on equations in *divergence form* that is on PDEs of the form

$$u_t - \frac{\partial}{\partial x}[a^{\text{div}}(t,x,u(t,x),u_x(t,x))] + b^{\text{div}}(t,x,u(t,x),u_x(t,x)) = 0$$

To write the PDE (4.33) in divergence form we have to choose

$$\begin{aligned} a^{\text{div}}(\rho,t,x,u,q) &:= \int_0^q \bar{a}(\rho,t,x,u,\tau)d\tau \\ b^{\text{div}}(\rho,t,x,u,q) &:= -\bar{b}(\rho,t,x,u,q)\cdot q + \frac{\partial a^{\text{div}}}{\partial x}(\rho,t,x,u,q) + \frac{\partial a^{\text{div}}}{\partial u}(\rho,t,x,u,q)\cdot q\,. \end{aligned}$$

To prove the statement on existence of solutions we now check that the hypothesis of (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.8.1) are satisfied.

ad a): This hypothesis is directly implied by Assumption (A.3).

ad b): Here we have for $A(\rho,t,x,u,q)$ defined in (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Chapter 5, equation (8.5))

$$A(\rho,t,x,u,q) := b^{\text{div}}(\rho,t,x,u,q) - \frac{\partial a^{\text{div}}}{\partial x}(\rho,t,x,u,q) - \frac{\partial a^{\text{div}}}{\partial u}(\rho,t,x,u,q)\cdot q = -\bar{b}(\rho,t,x,u,q)\cdot q$$

and hence $A(\rho,t,x,u,0)=0$.

ad c) To verify this hypothesis we have to work a little harder. We have to check condition b) of (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.6.1). Suppose we want to prove existence for some $\rho \in [0,1)$. Let us first fix some $1 > \rho_0 \geq \rho$. Defining $\tilde{K}_1$ by

$$\tilde{K}_1 := \sup\{|\bar{a}(\rho,t,x,u,q)|,\ \rho\in[0,\rho_0], t\in[0,T], x\in\mathbb{R},\ u\in I^g,\ q\in\mathbb{R}\} \tag{A.43}$$

we get immediately from the first inequality in Assumption (A.4) that $\tilde{K}_1 < \infty$. Here I^g denotes the interval $[\min_{f>0} g(f), \max_{f>0} g(f)]$. Moreover, we have

$$\tilde{K}_0 \leq \bar{a}(\rho,t,u,q) \leq \tilde{K}_1\,,$$

where $\tilde{K}_0$ is defined in (A.42). This proves the first part of the condition. The following estimates show that the second half is fulfilled, too:

$$|a^{\text{div}}(\rho,t,x,u,q)|\cdot(1+|q|) \leq |q|\cdot\tilde{K}_1\cdot(1+|q|) \leq \tilde{K}_1\cdot(1+|q|)^2$$

Now, using Assumption (A.4) it is easily shown that the following constants are finite:

$$\tilde{K}_2 := \sup\left\{\left|\frac{\partial a^{\text{div}}}{\partial u}(\rho,t,x,u,q)\right|,\ \rho\in[0,\rho_0], t\in[0,T], x\in\mathbb{R}, u\in I^g,\ q\in\mathbb{R}\right\} \tag{A.44}$$

$$\tilde{K}_3 := \sup\left\{\left|\frac{\partial a^{\text{div}}}{\partial x}(\rho,t,x,u,q)\right|,\ \rho\in[0,\rho_0], t\in[0,T], x\in\mathbb{R}, u\in I^g,\ q\in\mathbb{R}\right\} \tag{A.45}$$

$$\tilde{K}_4 := \sup\{|\bar{b}(\rho,t,x,u,q)|,\ \rho\in[0,\rho_0], t\in[0,T], x\in\mathbb{R}, u\in I^g,\ q\in\mathbb{R}\} \tag{A.46}$$

These constants depend of course on ρ_0, on the "cutoff-level" M in (A.41) and on the value of the constants in Assumption (A.4). Now we get

$$\begin{aligned}
\left|\frac{\partial}{\partial u}a^{\text{div}}(\rho,t,x,u,q)\cdot(1+|q|)\right| &\leq |q|\cdot\tilde{K}_2\cdot(1+|q|)\leq\tilde{K}_2\cdot(1+|q|)^2\\
\left|\frac{\partial}{\partial x}a^{\text{div}}(\rho,t,x,u,q)\right| &\leq |q|\cdot\tilde{K}_3\leq\tilde{K}_3\cdot(1+|q|)\text{ and}\\
|b^{\text{div}}(\rho,t,x,u,q)| &\leq \left(|\bar{b}(\rho,t,x,u,q)|+|\frac{\partial a^{\text{div}}}{\partial u}(\rho,t,x,u,q)|\right)\cdot|q|+|\frac{\partial a^{\text{div}}}{\partial x}(\rho,t,x,u,q)|\\
&\leq (\tilde{K}_4+\tilde{K}_2+\tilde{K}_3)\cdot(1+|q|)^2
\end{aligned}$$

Since the above estimates are valid for all $x\in\mathbb{R}$ we get $u\in H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$ which proves the first statement of the proposition. As explained in (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, p 451) the norm

$$\|u_x\|_T := \sup\{|u_x(t,x)|,\quad x\in\mathbb{R}, t\in[0,T]\}$$

is bounded by some constant K depending only on the Hölder norm of the initial values $\tilde{g}$ and the constants $\tilde{K}_0,\ldots,\tilde{K}_4$ from the above estimates. As these are valid for all $\rho\in[0,\rho_0]$ the second claim follows.

Uniqueness follows immediately from (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.6.1), since the coefficients of the PDE (4.33) are smooth functions. Hence for every $\rho\in[0,1)$ these functions and their derivatives are bounded on every the compact set of the form $\{(t,x,u,q),\ t\in[0,T], |x|<N, u\in I^g, |q|\leq K\}$, where K is the bound on u_x established in (*ii*). □

Proof of Theorem 4.2

ad (*i*): It is enough to prove existence of a solution to the initial value problem (4.29), (4.30). Consider first the case $\psi_\alpha\geq 0$. By Proposition 4.3 for every $\rho\in[0,1)$ the restricted Cauchy problem (4.33), (4.30) has a solution from $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$. Moreover for every $\rho_0\in[0,1)$ there is a constant $K=K(\rho_0,M)$ depending[8] only on ρ_0 and the "cutoff level" M from (A.41) such that for all $\rho\in[0,\rho_0]$ the norm of the derivative u_x of the solution u to (4.33), (4.30) is bounded by K. Moreover by Proposition 4.4 we have $u_x\geq 0$. Hence whenever

$$\rho\leq\bar{\rho}:=\sup\left\{\min\left\{\rho_0,\frac{M}{K(\rho_0,M)}\right\},\quad M>0,\rho_0\in[0,1]\right\}$$

the "constraints" of equation (A.41) are not binding such that u is also a solution of the unrestricted PDE (4.29). If $\psi_\alpha<0$ the argument is analogous. However now we have to choose $\bar{\rho}$ small enough to ensure that moreover the argument of c_1 takes its values in $[1-\varepsilon,M]$ for some small positive constants ε and some large M. As $\psi_\alpha/(f\cdot\psi_f)$ is bounded according to Assumption (A.4), this can be done.

ad (*ii*) This statement follows directly from Proposition 4.4.

[8] K depends on M and ρ_0 via the constants $\tilde{K}_i$ defined in the proof of Proposition 4.3.

ad (iii): Uniqueness for the terminal value problem for $\phi(t, f)$ is equivalent to uniqueness for the initial value problem for $u(t, x)$. Now suppose that u^1 and u^2 are both solutions belonging to $H^{2+\beta,1+\beta/2}([0, T] \times \mathbb{R})$ for some $\beta > 0$. It follows from condition (4.27) that for M large enough both functions solve also the restricted PDE (4.33). Hence the claim follows from Proposition 4.3. $\square$

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