Risk-minimizing Hedging Strategies under Restricted Information: The Case of Stochastic Volatility Models Observable only at Discrete Random Times

Rüdiger Frey
Swiss Banking Institute
University of Zürich
Plattenstr 14
CH-8032 Zürich
freyr@isb.unizh.ch*

Wolfgang J. Runggaldier
Dipartimento di Matematica
Pura ed Applicata
Universitá di Padova, Via Belzoni 7
I-35131-Padova, Italy
runggal@math.unipd.it

July 18, 1999

Abstract

We consider a market where the price of the risky asset follows a stochastic volatility model, but can be observed only at discrete random time points. We determine a local risk minimizing hedging strategy, assuming that the information of the agent is restricted to the observations of the price at its random jump times. Stochastic filtering also comes into play when computing the hedging strategy in the given situation of restricted information.

Key Words: Stochastic volatility, discontinuous prices, hedging under restricted information, risk minimizing hedging strategies, stochastic filtering, marked point processes.

1 Introduction

In this paper we consider a market with a risky and a nonrisky asset. The price of the risky asset follows a stochastic volatility model, where the volatility is influenced by some latent process X. Stochastic volatility models, which were developed to overcome some of the empirical deficiencies of the classical Black Scholes model, have received a lot of interest in the financial literature; see e.g. [11] for a survey of option hedging in these models.

*This work was done while the first author was at the Department of Mathematics, ETH Zurich. Financial support from UBS is gratefully acknowledged.

We consider the problem of hedging a derivative contract in this market from the viewpoint of an agent who observes the precise value of the stock price S_t only at discrete, random points in time $T_1 < T_2 < \cdots$. This departure from the usual modelling approach in Finance, where one assumes that prices are monitored on a continuous basis, makes sense from an economic viewpoint: as soon as agents incur some cost or have to invest some (small) effort in order to obtain accurate and up-to-date price information, it is reasonable to assume that they acquire precise price information only at certain discrete points in in time. This implies in particular, that they do not have precise information about the current volatility level.

The main purpose of this paper is to describe an approach to determine a hedging strategy for an agent who has only incomplete price observations. Clearly, any reasonable strategy for this agent has to depend in some way on the unobservable latent state process. We are thus in presence not only of an incomplete market situation, but also of partial information. We show that the criterion of (local) risk minimization is particularly appropriate to deal with this situation. Following a line of attack analogous to [8], where observable prices are modelled as diffusions, we proceed along two steps : first we determine a risk minimizing strategy assuming full knowledge also of the latent state process. Then, in line with [19] (see also [6]), we obtain the risk minimizing hedging strategy under partial information by "projecting" the full information strategy onto the subfiltration describing the available partial information that comes from observing the prices (or, equivalently, their logarithms) at the discrete random times where a trade occurs. In order that this two-step procedure is applicable, we model the price of the risky asset directly under a martingale measure.

To actually compute the projections onto the subfiltration, an important tool is the conditional distribution of the latent state process, given the available price observations. This leads to a nonlinear filtering problem with marked point process observations, that we study in the companion paper [12]. In the last section we recall the main features of the solution of this filtering problem and show how it can be applied to our setting, thus leading to a complete solution to the given hedging problem.

2 The model

Consider some underlying filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$ and some terminal date T. We are interested in the following three models for the dynamics of the latent process X which is assumed to influence the asset price volatility in our setup. In economic terms the process X can be interpreted as the rate at which private/insider information is absorbed by the market.

A1) X is a RCLL piecewise constant interpolation of a discrete time, finite state Markov chain over an equispaced time grid with step $\Delta > 0$. The state space is $E_M := \{x_1, \cdots, x_M\}$ and the transition probability matrix is $\Pi = \{p_{ij}\}_{i,j=1,\cdots,M}$.

A2) X is a continuous-time finite state Markov chain with state space $E_M := \{x_1, \cdots, x_M\}$ and generator matrix $R = \{r_{ij}\}_{i,j=1,\cdots,M}$.

A3) X is a diffusion, i.e. for a Wiener process $\{w_t^1\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ we have

$$dX_t = a(t, X_t)dt + \eta(t, X_t)dw_t^1 \tag{1}$$

with coefficients such that there exists a unique weak sense solution.

We assume that the asset price S follows a stochastic volatility model of the form

$$dS_t = S_t\sqrt{v(t, S_t, X_t)}\, dw_t^0 \tag{2}$$

for a second Brownian motion w^0, which is independent of the filtration generated by X. Here the function $v : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous with values $v(t, S, x) \in [\underline{v}, \overline{v}]$, $\forall t \in [0, T]$, $S \in \mathbb{R}^+$, $x \in \mathbb{R}$ where $0 < \underline{v} \leq \overline{v} < \infty$. The initial condition S_0 is supposed to be deterministically given.

Besides S, there is a risk-free asset B traded in this market (a bond or money market account). For simplicity we take the price $B_t \equiv 1$, i.e. S represents the forward price of the risky asset. Note that under this assumption model (2) implies that the measure P is a martingale measure for S.

As mentioned in the Introduction, we consider the problem of hedging a derivative contract in this market from the viewpoint of an agent who observes the precise value of the stock price S_t only at discrete, random points in time $T_1 < T_2 < \cdots$. The random times T_n are modelled as jump times of some point process $N = (N_t)$, whose $\{\mathcal{F}_t\}-$intensity $\lambda_t = \lambda(t, X_{t-})$ depends on X_t. More precisely, $\lambda : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing in x with $\lambda(t, x) \in [\underline{\lambda}, \overline{\lambda}], > \forall t \in [0, T]$, $x \in \mathbb{R}$ where $0 < \underline{\lambda} \leq \overline{\lambda} < \infty$. Assuming that λ is nondecreasing makes sense from an economic viewpoint as the agent is likely to monitor the market more frequently in periods where the market is very active or where a lot of new economic information reaches the market (high volatility periods). The time dependence in λ and in v above is introduced to allow for the incorporation of seasonal effects, which are typical for high frequency data. For more information about qualitative properties of high frequency data we refer the reader to [13].

Since observing S_t or its logarithmic value $L_t = \log S_t$ is equivalent, we assume that the information available to our economic agent comes from observing L_t at the random times T_n. Notice that T_{N_t} is the time of the last jump of N prior to t. The price information available to our agent can therefore be summarized by an information process Y defined via

$$Y_t := L_{T_{N_t}} \,. \tag{3}$$

The process (Y_t) is thus a marked point process (see [4] for the terminology); moreover, the jump times of Y and N coincide $P-$a.s. Marked point processes have recently become popular as models for asset price dynamics; see e.g. [1], [17] or [18].

The information available to our economic agent can be modelled by the subfiltration $\{\mathcal{F}_t^Y\}$, generated by the process Y, which can equivalently be expressed as

$$\mathcal{F}_t^Y = \{N_s,\, s \leq t\,;\, L_{T_i}\,,\, T_i \leq t\}; \tag{4}$$

we shall furthermore assume that, for $t = T$,

$$\mathcal{F}_T^Y = \{N_s,\, s \leq T\,;\, L_{T_i}\,,\, T_i \leq T\,;\, L_T\} \tag{5}$$

i.e., at the terminal date T the value of L_T (equivalently S_T) can be observed exactly.

For the filtering results in Section 6 we need an additional assumption on the point process N. To formulate this assumption in a mathematical precise manner we introduce the filtration $\{\mathcal{G}_t\}$ defined by $\mathcal{G}_t := \mathcal{F}_t^Y \vee \mathcal{F}_T^X$. Note that $\{\mathcal{G}_t\}$ contains information about all the future of the state variable process.

A4) N is a conditional Poisson process (Cox process), i.e. it admits the $P, \{\mathcal{G}_t\}$-intensity $\lambda(t, X_{t-})$.

Note that A4) is stronger than assuming that N is a point process with $P, \{\mathcal{F}_t\}$-intensity $\lambda(t, X_{t-})$; in particular A4) excludes the possibility that the jump-times of the processes N and X coincide.

3 Problem formulation

We shall consider the problem of hedging a contingent claim $H \in \mathcal{F}_T^S$, when the information of the economic agent is restricted to the filtration $\{\mathcal{F}_t^Y\}$. Since our market is incomplete even with continuous price observations, we have to choose some approach to hedging derivatives under incompleteness to determine hedging strategies. In this paper we shall use the criterion of risk minimization (see [9], [10]). Notice that, when working under a martingale measure, the criteria of local and remaining risk are equivalent. As we shall see, this (quadratic) criterion is particularly well suited to dealing with hedging under restricted information (in this context see [6], [19]). Although the results on risk minimizing hedging strategies are valid for general claims $H \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$, for actual computation we shall restrict ourselves to claims, whose payoff is a function $H(S_T)$ with $E\{(H(S_T))^2\} < \infty$.

A dynamic $\{\mathcal{F}_t\}$-*trading strategy* $(\xi, \eta) = \{(\xi_t, \eta_t),\ 0 \le t \le T\}$ is a rule to hold ξ_t units of the risky asset S and η_t units of B at time t. As usual we require ξ to be $\{\mathcal{F}_t\}$-predictable and η to be $\{\mathcal{F}_t\}$-adapted. The *value process* of this strategy is given by

$$V_t := V_t(\xi, \eta) := \xi_t X_t + \eta_t, \tag{6}$$

and the strategy is said to *hedge against* H if $V_T = H$. As we are working in an incomplete market context, our strategies will not necessarily be self-financing. We define the (cumulative) *cost process* of a trading strategy via

$$C_t := C_t(\xi, \eta) := V_t - \int_0^t \xi_s dS_s\,. \tag{7}$$

Note that a strategy is self-financing if and only if the cost process is constant.

In computing his strategy, our hedger has only the information contained in $\{\mathcal{F}_t^Y\}$ at his disposal. He is therefore restricted to the subclass of $\{\mathcal{F}_t^Y\}$-strategies where ξ is $\{\mathcal{F}_t^Y\}$-predictable and η is $\{\mathcal{F}_t^Y\}$-adapted. If moreover,

$$E\left\{\int_0^T \xi_t^2 v(t, S_t, X_t)\, S_t^2 ds\right\} < \infty \text{ and } E\left\{\sup\{|V_t(\xi,\eta)|,\ 0 \le t \le T\}^2\right\} < \infty\,, \tag{8}$$

a $\{\mathcal{F}_t^Y\}$-strategy is called $\{\mathcal{F}_t^Y\}$-*admissible*.

As in [9], [10] we take the conditional variance of the cost-process as measure of the risk of our strategy. The $\{\mathcal{F}_t^Y\}$-*risk process* $R^Y(\xi, \eta)$ of an $\{\mathcal{F}_t^Y\}$-admissible strategy is defined as

$$R_t^Y(\xi, \eta) := E\{(C_T(\xi, \eta) - C_t(\xi, \eta))^2 | \mathcal{F}_t^Y\}\,. \tag{9}$$

Finally we can give

Definition 3.1 *Given a contingent claim $H(S_T) \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$. An $\{\mathcal{F}_t^Y\}$-admissible strategy (ξ, η) which hedges against H is called $\{\mathcal{F}_t^Y\}$-risk minimizing if $R_t^Y(\xi, \eta) \le R_t^Y(\tilde{\xi}, \tilde{\eta})$ P-a.s. for all $t \in [0, T]$ and any $\{\mathcal{F}_t^Y\}$-admissible strategy $(\tilde{\xi}, \tilde{\eta})$ which hedges against H.*

As shown in [6] and [19], to compute $\{\mathcal{F}_t^Y\}$- risk-minimizing strategies, one can proceed in two steps. In a first step one determines the risk-minimizing strategy under "full information", i.e. for an agent who is able to use $\{\mathcal{F}_t\}$-trading strategies. The $\{\mathcal{F}_t^Y\}$-risk-minimizing strategy can then be computed by "projecting" the $\{\mathcal{F}_t\}$-risk-minimizing strategy onto the set of $\{\mathcal{F}_t^Y\}$-admissible strategies.

4 Risk minimizing hedging strategies under full information

We shall now determine a risk minimizing hedging strategy under full information $\{\mathcal{F}_t\}$. For this purpose define the $P-$martingale

$$g(t, S_t, X_t) := E\{H(S_T) \,|\, \mathcal{F}_t\} \tag{10}$$

where the definition is justified by the Markov property of (S_t, X_t). We have

Proposition 4.1 *Under sufficient regularity, the function $g(t,S,x)$ in (10) satisfies the following PDE's (Kolmogorov backwards equations) in $[0,T]\times \mathbb{R}^+ \times E_M$ respectively in $[0,T]\times\mathbb{R}^+\times\mathbb{R}$.*

a) *under assumption A1) for X_t :*

$$\begin{array}{l} g_t(t,S,x) + \frac{1}{2}v(t,S,x)S^2 g_{SS}(t,S,x) + \sum_{k=0}^{\infty} 1_{\{t=k\Delta\}}\left[\Pi - I\right] g(t,S,x) = 0 \\ g(T,S,x) = H(S) \end{array} \tag{11}$$

where $[\Pi - I]\, g(t,S,x) = \sum_{j\neq x} p_{x,j}\left[g(t,S,j) - g(t,S,x)\right]$ with $p_{x,j}$ the transition probabilities in A1).

b) *Under assumption A2) for X_t :*

$$\begin{array}{l} g_t(t,S,x) + \frac{1}{2}v(t,S,x)S^2 g_{SS}(t,S,x) + Rg(t,S,x) = 0 \\ g(T,S,x) = H(S) \end{array} \tag{12}$$

where $Rg(t,S,x) = \sum_{j\neq x} r_{x,j}\left[g(t,S,j) - g(t,S,x)\right]$.

c) *Under assumption A3) for X_t (and in $[0,T]\times\mathbb{R}^+\times\mathbb{R}$) :*

$$\begin{array}{l} g_t(t,S,x) + g_X(t,S,x)\, a(t,x) + \frac{1}{2}g_{SS}(t,S,x)v(t,S,x)S^2 + \frac{1}{2}g_{XX}(t,S,x)\eta^2(t,x)] = 0 \\ g(T,S,x) = H(S)\,. \end{array} \tag{13}$$

Furthermore, in all three cases the risk minimizing hedging strategy under full information $(\xi^{\mathcal{F}}, \eta^{\mathcal{F}})$ is given by :

$$\begin{array}{rcl} \xi_t^{\mathcal{F}} &=& g_S(t, S_t, X_{t-}) \\ \eta_t^{\mathcal{F}} &=& g(t,S_t,X_t) - \xi_t^{\mathcal{F}} S_t\,. \end{array} \tag{14}$$

Proof : From [10] or [9] we know that, if one can determine the *Kunita Watanabe decomposition (KW)*

$$H(S_T) = H_0 + \int_0^T \xi_t^H dS_t + M_T^H \tag{15}$$

of the claim H, where M^H is a $P-$martingale orthogonal to the $P-$martingale S, then $\xi_t^{\mathcal{F}} = \xi_t^H$ and M^H determines the cost process.

For this purpose let us first consider the process X_t according to assumptions A1) or A2). Using Ito's formula, we may write

$$\begin{array}{rcl} H(S_T) &=& g(T,S_T,X_T) = g(0,S_0,X_0) + \int_0^T g_S(t,S_t,X_{t-})dS_t \\ &+& \int_0^T \left[g_t(t,S_t,X_{t-}) + \frac{1}{2}v(t,S_t,X_{t-})S_t^2 g_{SS}(t,S_t,X_{t-})\right] dt \\ &+& \sum_{t\leq T}\left[g(t,S_t,X_t) - g(t,S_t,X_{t-})\right]\,. \end{array} \tag{16}$$

Under A1) we now have that

$$\begin{aligned}&\textstyle\sum_{s\leq t=n\Delta}\left[g(s,S_s,X_s)-g(s,S_s,X_{s-})\right]\\&\qquad=M_t^{(1)}+\sum_{k=0}^{n}(\Pi-I)\,g(k\Delta,S_{k\Delta},X_{(k-1)\Delta})\end{aligned}\tag{17}$$

where $M_t^{(1)}$ is a $(P,\mathcal{F})$-martingale with piecewise constant trajectories and jump-times equal to $k\Delta$, $k=1,2,\ldots,K$. Following now a rather standard approach (see e.g. [11]), we have that all predictable finite variation terms have to vanish, since $g(t,S_t,X_t)$ is a $(P,\{\mathcal{F}_t\})-$ martingale. This leads on one hand to (11), on the other hand to the representation

$$H(S_T)=g(0,S_0,X_0)+\int_0^T g_S(t,S_t,X_{t-})\,dS_t+M_T^{(1)}\,,\tag{18}$$

which is the required KW-decomposition of $H(S_T)$, as the finite variation martingale $M^{(1)}$ is orthogonal to the $P-$martingale S.

Analogous considerations apply under A2). It is well-known that the process

$$M_t^{(2)}:=\sum_{s\leq t}g(s,S_s,X_s)-g(s,S_s,X_{s-})-\int_0^t Rg(s,S_s,X_{s-})ds\tag{19}$$

is a martingale. Hence, under (12) we have the representation

$$H(S_T)=g(0,S_0,X_0)+\int_0^T g_S(t,S_t,X_{t-})\,dS_t+M_T^{(2)}\,,\tag{20}$$

which proves the claim, as S and $M^{(2)}$ are again orthogonal.

Coming to A3) and using always Ito's formula as well as the equations (2) and (1) for S_t and X_t respectively, we may write

$$\begin{aligned}H(S_T)\;&=g(0,S_0,X_0)+\textstyle\int_0^T\left[g_t(t,S_t,X_t)+g_X(t,S_t,X_t)\,a(t,X_t)\right]\,dt\\&\quad+\textstyle\int_0^T\left[\frac{1}{2}g_{SS}(t,S_t,X_t)v(t,S_t,X_t)S_t^2+\frac{1}{2}g_{XX}(t,S_t,X_t)\eta^2(t,X_t)\right]dt\\&\quad+\textstyle\int_0^T g_S(t,S_t,X_t)dS_t+\int_0^T g_X(t,S_t,X_t)\eta(t,X_t)\,dw_t^1\,.\end{aligned}\tag{21}$$

Again, since $g(t,S_t,X_t)$ is a $P-$martingale, the (predictable) finite-variation terms have to vanish, which leads to (13). Furthermore, $\{w_t^1\}$ being independent of $\{w_t^0\}$, the last term on the right in (21) is a $P-$martingale, orthogonal to S_t, which proves c).

Q.E.D.

To compute the hedging strategy and its value, we need thus to compute $g(t,S,x)$ and for this we may either compute the expectation in (10) or solve the PDE's in (11)-(13). Exact (analytical) solutions are difficult to obtain and so one is led to search for numerical/approximate solutions:

- To solve numerically the PDE's in (11)-(13) one usually employs finite difference methods; see e.g. ch 10.4 in [7] or [20] for a general account on these methods and [3] for a method which is particularly well suited for problems with a multidimensional state space.

- The numerical computation of the expectation in (10) can be obtained by a Monte Carlo simulation approach (see e.g. [2]) combined with discretization schemes for SDE's (see e.g. [14]).

5 Risk minimizing hedging strategies under incomplete information

We now go back to the original situation where the hedger is restricted to the information contained in $\{\mathcal{F}_t^Y\}$ as defined in (4), (5). Applying Theorem 2.5 and in particular relation (3.3) of [19] we get that a $\{\mathcal{F}_t^Y\}$-risk-minimizing strategy (ξ^*, η^*) can be computed from $(\xi^{\mathcal{F}}, \eta^{\mathcal{F}})$ as follows:

$$\begin{aligned} \xi_t^* &= E\left\{v(t, S_t, X_t)\, S_t^2 \xi_t^{\mathcal{F}} | \mathcal{F}_{t-}^Y\right\} \Big/ E\left\{v(t, S_t, X_t)\, S_t^2 | \mathcal{F}_{t-}^Y\right\} &(22)\\ \eta_t^* &= E\left\{H(S_T) - \xi_t^* S_t | \mathcal{F}_t^Y\right\} ; &(23) \end{aligned}$$

see also Theorem 1 of [6] for a related result.

According to our model, the hedger does not receive "significant new information" between the jump-times of N. Hence it is legitimate to assume that he updates his portfolio only immediately after the jumps of N, in particular, if the time between the jumps is small. More precisely, we assume that our hedger follows a piecewise constant strategy $(\tilde{\xi}^*, \tilde{\eta}^*)$ given by

$$\tilde{\xi}_t^* := \left[\lim_{s \downarrow T_i} \xi_s^*\right] 1_{(T_i, T_{i+1}]}(t) \text{ and } \tilde{\eta}_t^* := \left[\lim_{s \downarrow T_i} \eta_s^*\right] 1_{[T_i, T_{i+1})}(t)\,. \qquad (24)$$

Note that at the jump-times of N the value of S is observable. Hence we have for $t \in (T_i, T_{i+1})$

$$\begin{aligned} \tilde{\xi}_t^* &= E\left\{v(T_i, S_{T_i}, X_{T_i}) \xi_{T_i}^{\mathcal{F}} | \mathcal{F}_{T_i}^Y\right\} \Big/ E\left\{v(T_i, S_{T_i}, X_{T_i}) | \mathcal{F}_{T_i}^Y\right\} \\ &= E\left\{v(T_i, S_{T_i}, X_{T_i}) g_S(T_i, S_{T_i}, X_{T_i}) | \mathcal{F}_{T_i}^Y\right\} \Big/ E\left\{v(T_i, S_{T_i}, X_{T_i}) | \mathcal{F}_{T_i}^Y\right\} . \end{aligned} \qquad (25)$$

At the jump times T_i we define the strategy so that $\tilde{\xi}^*$ becomes left-continuous and $\tilde{\eta}^*$ right-continuous.

In the sequel we make also the following assumption :

A5): The function g defined in (10) satisfies the Kolmogorov backward equations (11)-(13). Moreover, for all fixed values of t, S, we have that $g(t, S, \cdot)$ and its derivative $g_S(t, S, \cdot)$ are bounded in x.

Since $v(\cdot)$ is assumed to be continuous and bounded, under assumption A5) the required conditional expectations in (25) can be computed whenever one can compute a conditional expectation of the form $E\left\{F(X_{T_i})|\mathcal{F}^Y_{T_i}\right\}$ with $F(\cdot)$ continuous and bounded. This will be the subject of the next section.

6 Computation of the conditional expectations via nonlinear stochastic filtering techniques

The purpose of this section is to present a method to compute conditional expectations of the form $E\left\{F(X_{T_i})|\mathcal{F}^Y_{T_i}\right\}$ where $F(\cdot)$ is continuous and bounded, X_t is the unobserved process satisfying one of the assumptions A1)-A3), T_i are the jump times of the process N, at which the price S_t of the risky asset (or, equivalently, its logarithmic value) can be observed, $\{\mathcal{F}^Y_t\}$ is the observation filtration as defined in (4), (5). This is a typical nonlinear stochastic filtering problem.

It is shown in [12] that for our model there exists a measure Q on $(\Omega, \mathcal{F}_T)$, equivalent to P, under which (X_t) and (Y_t) are independent. Denote by $\Lambda_t := dQ_{|\mathcal{F}_t} \Big/ dP_{|\mathcal{F}_t}$ the Radon-Nikodym derivative of Q with respect to P. By the exponential formula from Lebesgue-Stieltjes calculus we get that Λ_t is a functional of the trajectories of X and Y up to time t, i.e. $\Lambda_t = \Lambda_t(X^t, Y^t)$. By the so-called Kallianpur-Striebel formula (see e.g. [15]), which is related to Bayes' formula, one has

$$E^P\{F(X_t)|\mathcal{F}^Y_t\} = \frac{E^Q\{F(X_t)\Lambda_t^{-1}|\mathcal{F}^Y_t\}}{E^Q\{\Lambda_t^{-1}|\mathcal{F}^Y_t\}}\,. \tag{26}$$

The advantage of this formula is that it reduces the computation of the desired conditional expectation on the left to two expectations on the right which, due to the independence of (X_t) and (Y_t) under Q, are (roughly speaking) just ordinary expectations of a functional of the process (X_t), in which Y_t is fixed at the observed values. The measure Q can, furthermore, be chosen so that (see again [12]) the distribution of (X_t) under Q remains the same as under P. We now explain how this measure transformation approach leads to (approximative) recursion formulas for the conditional distribution of X_t given observations up to time t. We start with the case where X satisfies A1).

6.1 Case of assumption A1)

Putting

$$V_t(Y;F) := E^Q\{F(X_t)\Lambda_t^{-1}|\mathcal{F}_t^Y\} \tag{27}$$

let, for $x_i \in E_M \quad (i = 1,\cdots,M)$

$$q_t^Y(x_i) := V_t\left(Y;\delta_{\{x_i\}}\right) = E^Q\left\{1_{\{X_t=x_i\}}\Lambda_t^{-1}(X^t,y^t)\right\} \tag{28}$$

where y^t denotes the observed trajectory of (Y_t) from time 0 up to and including t. With (28) we may write (27) as

$$V_t(Y;F) = \sum_{i=1}^{M} F(x_i)\, q_t^Y(x_i) \tag{29}$$

so that (26) becomes (since (X_t) has the same distribution under P and Q, we can now drop the superscript in the expectation symbol)

$$E\{F(X_t)|\mathcal{F}_t^Y\} = \frac{\sum_{i=1}^{M} F(x_i)\, q_t^Y(x_i)}{\sum_{i=1}^{M} q_t^Y(x_i)} \tag{30}$$

showing that $p_t^Y(x_i) := q_t^Y(x_i)\big/\sum_{j=1}^{M} q_t^Y(x_j)$ is the conditional distribution (*filter distribution*) of the unobserved process X_t, given past and present observations of (Y_t). Consequently, $q_t^Y(x_i)$ can be viewed as unnormalized filter distribution.

Our filtering problem is thus solved, whenever we are able to compute the vector $q_t^Y := \left(q_t^Y(x_1),\cdots,q_t^Y(x_M)\right)$ of unnormalized filter distributions, which we need for all values of $t = T_i$. In the next Proposition, assuming that the function v in (2) does not depend on S, we present for q_t^Y a recursion over the discrete time grid with step $\Delta > 0$ (see assumption A1)). For this purpose we put $q_k^Y := q_{k\Delta}^Y$.

Proposition 6.1 *Suppose that, in addition to the given assumptions, the function v in (2) does not depend on S. Then the vector q_k^Y of unnormalized filter distributions with components as given in (28) satisfies the following recursion*

$$q_{k+1}^Y = \left(\Pi^T \cdot E^k\right)\cdot q_k^Y \tag{31}$$

where Π^T is the transpose of the transition probability matrix of X_t as given in assumption A1) and where

$$E^k := diag\left(E_j^k;\ j = 1,\cdots,M\right) \tag{32}$$

with

$$E_j^k := \exp\left\{\Delta - \textstyle\int_{k\Delta}^{(k+1)\Delta}\lambda(t,x_j)\,dt + \displaystyle\sum_{k\Delta<T_m\le(k+1)\Delta}\left[\frac{1}{2}\log\left(\frac{\underline{v}(T_m-T_{m-1})}{\int_{T_{m-1}}^{T_m} v(t,x_j)\,dt}\right)\right.\right.$$
$$\left.\left.+\ \tfrac{1}{2}\left(\textstyle\int_{T_{m-1}}^{T_m} v(t,x_j)dt - \underline{v}(T_m-T_{m-1})\right)\left(\frac{(L_m-L_{m-1})^2}{\underline{v}(T_m-T_{m-1})\int_{T_{m-1}}^{T_m} v(t,x_j)dt} - \frac{1}{4}\right) + \log\lambda(T_m,x_j)\right]\right\} \tag{33}$$

Here $v(t,x)$ is as in (2) (independent of S), $\underline{v}$ is the lower bound on $v(t,x)$, $\lambda(t,x)$ the intensity of the point process N, T_m the observed jump times of N in $(k\Delta,(k+1)\Delta]$ (if no jump of N is observed in $(k\Delta,(k+1)\Delta]$, the sum in the right of (33) drops out), L_m is the value of $L = \log S$ as observed at $t = T_m$, and x_j is the $j-th$ element in E_M.

Remarks :

1. Recursion formulae of the form (31) are particularly useful if new observations arrive very frequently. In a financial context this quick arrival of new information is typical for High Frequency Data.

2. If the functions v and λ do not depend explicitly on time, the expression for E_j^k simplifies considerably and E_j^k itself becomes time independent.

3. In the special case when one neglects the information coming from the jump-heights of Y, which corresponds to assuming $v(t,S,x) \equiv \underline{v} = \overline{v}$, we obtain a known formula (see e.g. [5]) for pure jump observations.

As a detailed mathematical proof is given in the companion paper [12], we discuss only the basic idea behind the result. To derive the recursion formula (31) one conditions on $\mathcal{F}_{k\Delta}^X$ in (28), evaluated at $t = (k+1)\Delta$. The result then follows from the fact that the multiplicative increment of the likelihood has a particular form as X is piecewise constant: we have in fact

$$\frac{\Lambda_{(k+1)\Delta}^{-1}\left(X^{(k+1)\Delta}, y^{(k+1)\Delta}\right)}{\Lambda_{k\Delta}^{-1}\left(X^{k\Delta}, y^{k\Delta}\right)} = \sum_{j=1}^{M} 1_{\{X_{(k)\Delta}=x_j\}} E_j^k \,.$$

Notice also that, in the companion paper [12], the process S_t satisfies (2) with a drift term $-\frac{1}{2}S_t^2 v(t,S_t,X_t)$ (in [12] it is the log-price L_t that is a $P-$martingale). This implies a slight change in the expression for the Radon-Nikodym derivative and, consequently, also in the expression for E_j^k (the only difference being that, here, we have the additional term $-1/4$ in the right hand side of (33)).

6.2 Case of assumptions A2) and A3)

In both these cases we still have formula (26), but the functional $V_t(Y;F)$ is now the expectation of a functional of a continuous time process, and this makes it considerably more difficult to compute, unless one uses a Monte Carlo type approach. In particular, as soon as the state space of X is infinite, the filtering problem does not admit a recursive solution. Using the independence under Q of (X_t) and (Y_t), it can be shown (see Proposition 3.2 in [12]) that $V_t(Y;F)$ in (27) can be approximated arbitrarily closely by the sequence

$$V_t^n(Y;F) := E\left\{F(X_t^n)\Lambda_t^{-1}((X^n)^t, y^t)\right\} \tag{34}$$

where $X^n = (X_t^n)$ is a sequence of discrete time and finite state Markov chains, whose piecewise constant time interpolations converge weakly on the Skorokhod space D to the original process $X = (X_t)$. Such a sequence can be constructed for X_t satisfying both A2) and A3), since under these assumptions it does not admit deterministic times of discontinuity (i.e. $P\{X_t \neq X_{t-}\} = 0$). Various ways to construct such a sequence can be found in the existing literature (see e.g. [16]). For the computation of the approximating functional $V_t^n(Y;F)$ one can then use the results of the previous subsection.

References

[1] O.E. Barndorff-Nielsen and N. Shephard, Aggregation and model construction for volatility models. Preprint, Center for Analytical Finance, University of Aarhus, 1998.

[2] P. Boyle, M. Broadie, and P.Glasserman, Monte Carlo Methods for Security Pricing, *Journal of Economic Dynamics and Control.*

[3] I. Craig and D. Sneyd, An alternating directions implicit scheme for parabolic equations with mixed derivatives , *Comput. Math. Appl.* 16, 341–350.

[4] P.Bremaud, *Point Processes and Queues : Martingale Dynamics*, Springer Verlag, New York, 1981.

[5] G.B.Di Masi, and W.J. Runggaldier, On approximation methods for nonlinear filtering, *In : S.K. Mitter and A.Moro (eds.), Nonlinear Filtering and Stochastic Control, Lecture Notes in Mathematics Vol. 972*, Springer Verlag Berlin, 249–259, 1982.

[6] G.B.Di Masi, E.Platen, and W.J. Runggaldier, Hedging of options under discrete observation on assets with stochastic volatility, *In: E.Bolthausen, M.Dozzi and F.Russo (eds.), Seminar on Stochastic Analysis, Random Fields and Applications, Progress in Probability Vol.36*, Birkhäuser Verlag, 359–364, 1995.

[7] D.Duffie, *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton, NJ, 1992.

[8] P. Fisher, E. Platen, and W. Runggaldier, Risk-minimizing hedging strategies under partial information. To appear in *Proceedings of the 1996 Monte Veritá Seminar on Stochastic Analysis, Random Fields and Applications*, Birkhäuser Verlag.

[9] H.Föllmer, and M. Schweizer, Hedging of contingent claims under incomplete information, *In : M.H.A. Davis and R.J. Elliott (eds.), Applied Stochastic Analysis, Stochastic Monographs, Vol.5*, Gordon & Breach, London/New York, 389–414, 1991.

[10] H.Föllmer, and D. Sondermann, Hedging of non-redundant contingent claims *In: W. Hildenbrand and A. Mas-Colell (eds.), Contributions to Mathematical Economics*, North Holland, 205–223, 1986.

[11] R. Frey, Derivative asset analysis in models with level-dependent and stochastic volatility. *CWI Quarterly, Amsterdam*, 10:1–34, 1997.

[12] R. Frey, and W.J. Runggaldier, A nonlinear filtering approach to volatility estimation with a view towards high frequency data. Submitted, available under `http://www.math.ethz.ch/~frey` (1999).

[13] D.M. Guillaume, M. Dacorogna, R. Davé, U. Müller, R. Olsen, and P. Pictet. From the bird's eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets. *Finance and Stochastics*, 1:95–129, 1997.

[14] N.Hofmann, E. Platen, and M. Schweizer, Option Pricing under Incompleteness and Stochastic Volatility, *Mathematical Finance* 2 : 153–187, 1992.

[15] G. Kallianpur, *Stochastic Filtering Theory*, Springer Verlag, New York, 1980.

[16] H.J.Kushner and P.G. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer Verlag 1993.

[17] L.C.G. Rogers and O. Zane, Designing models for high-frequency data. Preprint, University of Bath, 1998.

[18] T. Rydberg and N. Shephard, Dynamics of trade-by-trade price movements : decomposition and models. *CAF Working paper series no. 21. Aarhus School of Business.*

[19] M. Schweizer, Risk minimizing hedging strategies under restricted information. *Mathematical Finance*, 4:327–342, 1994.

[20] P.Wilmott, J. Dewynne and S. Howison, *Option Pricing, Mathematical Models and Computation*, Oxford Financial Press, Oxford, 1993.