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THE GENERALIZATION OF THE GESKE–FORMULA FOR COMPOUND OPTIONS TO STOCHASTIC INTEREST RATES IS NOT TRIVIAL – A NOTE

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Abstract

This note refers to the paper *"Changes of Numeraire, Changes of Probability Measure and Option Pricing"* by Geman et al. [GER95], in which an extension of the Geske–formula for compound options to the case of stochastic interest rates is proposed. We show that such an extension is not possible in general. However, we point out modifications of Geske's original problem in which closed formulas can still be obtained under stochastic interest rates. In particular we consider the case of an option on a futures–style option. Moreover, we sketch a numerical solution to Geske's original problem when interest rates are random.

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1. Why there is no simple Generalization of the Geske–Formula to Stochastic Interest Rates

In his 1979 paper *"The Valuation of Compound Options"* Geske [Ges79] studied the valuation by arbitrage of a call option on a call option on a stock in the framework of the Black–Scholes model. More precisely, if we denote by S_τ the price of the stock at time $\tau > 0$ and by $C_{T_2}(S_{T_2}, K_1, T_1)$ the price at time $0 < T_2 < T_1$ of an European Call option on the stock S with strike K_1 maturing at time T_1 Geske was interested in the time $0 < t < T_2$ value of the following payoff received at time T_2:

$$C^{co}_{T_2} \quad = \quad \left[C_{T_2}\left(S_{T_2}, K_1, T_1\right) - K_2\right]^+ . \tag{1}$$

He used the framework proposed by Black–Scholes, i.e. he essentially assumed the process of the stock price to be given by the solution to the SDE

$$dS_t \quad = \quad rS_t dt + \sigma S_t dW_t,$$

where r and σ are constants and W is a one dimensional standard Brownian motion. In this framework the price of a European call option at time τ on S with strike K and maturity T is given by

$$C_\tau(S_\tau, K, T) = S_\tau \mathcal{N}(d_1) - Ke^{-r(T-\tau)}\mathcal{N}(d_2) \qquad \text{where}$$

$$d_1 = \frac{\ln\left(\frac{S_T}{Ke^{-r(T-\tau)}}\right) + \frac{1}{2}\sigma^2(T-\tau)}{\sigma\sqrt{(T-\tau)}} \ \text{ and } \ d_2 = d_1 - \sigma\sqrt{T-\tau}$$

This is easily seen to be strictly monotonically increasing in S_τ from zero to infinity as S_τ goes from zero to infinity. Hence for any $K^* \in \mathbb{R}_+$ there is exactly one $S^*(K^*) \in \mathbb{R}_+$ such that $C_\tau(S^*, K, T) = K^*$. The existence of such an S^* is crucial for the derivation of the Geske formula because it allows the decision whether the compound call as defined in equation (1) is to be exercised or not at time T_2 to be reduced to the question of whether $S_{T_2} > S^*(K_2)$. Given this fact and using the law of iterated expectations, by absence of arbitrage (see [HP81]) the price at time $t < T_2$ of the compound call can be written as

$$\begin{aligned} C^{co}_t \quad = \quad E_t \Big[& \exp\{-r(T_2 - t)\} 1_{\{S_{T_2} > S^*(K_2)\}} \\ & \Big(E_{T_2}\Big[\exp -r(T_1 - T_2)(S_{T_1} - K_1) 1_{\{S_{T_1} > K_1\}}\Big] - K_2 \Big) \Big] . \end{aligned}$$

Using the law of iterated expectations and the change of measure technique it is now easy to derive the Geske formula under deterministic interest rates (see [DR92]).

In their paper *"Changes of Numeraire, Changes of Probability Measure and Option Pricing"* Geman et al. [GER95] claim that this formula can be generalized to the case of stochastic interest rates. They do not specify any particular model of the term structure of interest rates. All they require for their formula to hold is that the volatility of the stock price be deterministic. We show that under these general conditions their formula for the compound call under stochastic interest rates is wrong. In particular we argue that if a generalization of the Geske formula to stochastic interest rates was to be possible this would require unacceptably severe restrictions on the volatility of the stock price process and on its correlation with the price processes of zero coupon bonds.

In order to see this let us extend our above model for the stock price so that it encompasses an arbitrage free model of the term structure of interest rates with deterministic volatilities of forward rates and hence of zero coupon bond prices along the lines of Heath et al. [HJM92]. We assume that zero coupon bonds of all maturities $T \in [0, \bar{T}]$ are traded. For $t \leq T$ the price at t of the zero coupon bond maturing at T shall be denoted by $B(t,T)$. In an arbitrage free market in the absence of credit risk we must of course have that $B(T,T) \equiv 1$ for all $T \in [0, \bar{T}]$. If $B(t,T)$ is differentiable in the second argument the instantaneous forward rate $f(t,T)$ exists and is defined as $f(t,T) = -\frac{\partial}{\partial T} \ln B(t,T)$. If all forward rates exist we have the relation $B(t,T) = \exp(-\int_t^T f(t,s)ds)$. The process $(r_t)_{0\leq t} := (f(t,t))_{0\leq t}$ will be called the short rate process. Finally, by $\beta_{t,T} := \exp\{\int_t^T r_s ds\}$ we denote the accumulation factor or savings account. Since we are only interested in the pricing of derivatives by no-arbitrage arguments and we will presently assume that markets are complete it is legitimate to model the asset price dynamics directly under the risk-neutral measure P. Under this measure all non-dividend paying assets are (local) martingales after

As shown by Harrison and Pliska [HP81] market completeness is equivalent to uniqueness of the risk-neutral measure such that there is no ambiguity here.

discounting with the savings account, hence their instantaneous growth rate equals r_t. As shown by Heath et al. [HJM92] this implies that under P the drift of the forward rates can be expressed in terms of their volatilities. We make

Assumption 1.1 We are given a d–dimensional standard Brownian motion W_t^d *on a probability space* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le \bar{T}}, P\right)$. *The filtration is the augmented natural filtration of* W_t^d *and* $\mathcal{F} = \mathcal{F}_{\bar{T}}$.

1. *Let* $\sigma(t,T) : D := \{(t,T) \in \mathbb{R}_+^2 \mid 0 \le t \le T,\ 0 \le T \le \bar{T}\} \to \mathbb{R}^d$ *be a continuous function. Define* $\eta^B : \mathbb{R}_+^2 \to \mathbb{R}^d$ *by* $\eta^B(t,T) := -\int_t^T \sigma(t,s)ds$ *if* $(t,T) \in D$ *and* 0 *otherwise. We assume the following dynamics for the forward rates under* P*:*

$$df(t,T) = -\eta^B(t,T) \cdot \sigma(t,T)dt + \sigma(t,T)dW_t^d$$

2. *The stock price evolves according to*

$$dS_t = r_t S_t dt + S_t \eta_t^S dW_t^d$$

   where $\eta_t^S = \left(\eta_{1,t}^S, ..., \eta_{d,t}^S\right)$ *is a continuous* R^d*–valued function of time only.*

3. *There are d traded assets such that for all* $t \in [0,T]$ *the instantaneous covariance matrix of these assets is strictly positive.*

Remarks: As shown in [HJM92] this implies that $B(t,T)$ satisfies the SDE

$$dB(t,T) = r_t B(t,T)dt + B(t,T)\eta^B(t,T)dW_t^d\,. \tag{2}$$

The definition of the bond price "volatilities" implies that $\eta^B(t,\cdot)$ is differentiable in the second argument on D and that $\eta^B(T,T) \equiv 0$. Alternatively we could have started by specifying bond price volatilities η^B with these two properties and define bond prices as solution to the SDE (2), the short rate r_t being implicitely defined via $r_t = -\frac{\partial}{\partial T}|_{T=t} \ln B(t,T)$. The volatility of the forward rates is then given by $\sigma(t,T) = -\frac{\partial}{\partial T}\eta^B(t,T)$. This approach is taken in [EKR89], [EKMV92a]. Point 3 of this assumption implies that the market is complete, i.e. every integrable contingent claim adapted to the filtration generated by W^d can be replicated by a dynamic trading strategy, see section 6.I of [Duf92].

The Novikov criterion moreover implies that the discounted bond and stock price processes specified in Assumption 1.1 are strictly positive P-martingales (and not only local martingales). Theorem 1 of [GER95] therefore ensures that the price system we obtain by using one of these assets as new numeraire admits an equivalent martingale measure and is therefore arbitrage-free. As shown by Delbaen and Schachermayer in [DS95], if the new numeraire is only a local martingale it can happen that the price-system corresponding to this new numeraire allows arbitrage opportunities, even if the original price system admits an equivalent local martingale measure and is hence arbitrage-free. These authors also give a general characterization of those numeraires that preserve the no-arbitrage property. However, in our context we have no need of their general result.

In this framework, using the change of measure technique, it is easy to derive a generalized Black–Scholes formula for a European call option on the stock.

Proposition 1 Under Assumption 1.1 the generalization of the Black–Scholes formula to stochastic interest rates is given by

$$\tilde{C}_\tau\left(S_\tau, B(\tau,T), K, T\right) = S_\tau \mathcal{N}(d_1) - KB(\tau,T)\mathcal{N}(d_2) \quad \textit{where}$$

$$d_1 = \frac{\ln\left(\frac{S_T}{KB(\tau,T)}\right) + \frac{1}{2}\int_\tau^T |\eta_s^S - \eta^B(s,T)|^2 ds}{\sqrt{\int_\tau^T |\eta_s^S - \eta^B(s,T)|^2 ds}} \quad \textit{and } d_2 = d_1 - \sqrt{\int_\tau^T |\eta_s^S - \eta^B(s,T)|^2 ds}$$

We do not give a proof of this result here as Proposition 1 is a special case of Theorem 3.1 in [FS96].

The first observation to make about this formula is that it depends on the volatilities of the stock and the bond. Hence, the value of the compound call as in equation (1) cannot be independent of the volatility of the zero coupon bond maturing at T_1 as claimed in [GER95]. More important than this is the fact that the formula depends on two random variables, namely S_τ and $B(\tau,T)$, rather than one. This implies that under stochastic interest rates in contrast to deterministic ones it is not clear that the decision to exercise or not to exercise the compound call can still be reduced to a one–dimensional problem. The following theorem shows that a necessary condition for this to be possible is that in T_2 the logarithms of the stock and of the bond maturing at T_1 be perfectly correlated.

Theorem 1 Under Assumption 1.1 a necessary condition for the existence of a number $S^*(K_2) \in \mathbb{R}_+$ with the property that $\tilde{C}_{T_2}(S_{T_2}, B(T_2,T_1), K_1, T_1) > K_2 \Leftrightarrow S_{T_2} > S^*(K_2)$ is that the diffusion coefficients of the stock price process are given by

$$b\eta^S_{i,t} = \eta^B_i(t,T_1) - (1-b)\eta^B_i(t,T_2) \quad \forall t \in [0,T_2] \quad \text{and} \quad i = 1,\dots,d, \tag{3}$$

where b is a constant. As a consequence the $\mathcal{F}_{T_2}$-measurable random variables $\ln B(T_2,T_1)$ and $\ln S_{T_2}$ must be perfectly correlated.

Proof: A necessary condition for the existence of such a number S^* even under stochastic interest rates is that the pricing formula in Proposition 1 can be written as a function of S_{T_2} only. Hence, it must be possible to express $B(T_2,T_1)$ as a function of S_{T_2}. Now

$$\begin{aligned} S_{T_2} = S_{T_2}/B(T_2,T_2) &= g_S(T_2)\exp\left\{\int_0^{T_2} \eta^S_t - \eta^B(t,T_2)dW_t\right\} \\ B(T_2,T_1) = B(T_2,T_1)/B(T_2,T_2) &= g_B(T_2,T_1)\exp\left\{\int_0^{T_2} \eta^B(t,T_1) - \eta^B(t,T_2)dW_t\right\}, \end{aligned}$$

where g_S and g_B are constants. Now, if $B(T_2,T_1) = f(S_{T_2})$ there is clearly a function $\tilde{f}$ such that

$$\int_0^{T_2} \eta^B(t,T_1) - \eta^B(t,T_2)dW_t = \tilde{f}\left(\int_0^{T_2} \eta^S_t - \eta^B(t,T_2)dW_t\right).$$

Since the l.h.s. of the last equation and the argument of $\tilde{f}$ are normally distributed random variables with expected value equal to zero, and since normality is preserved only under affine transformations we have

$$\int_0^{T_2} \eta^B(t,T_1) - \eta^B(t,T_2)dW_t \stackrel{!}{=} b\int_0^{T_2} \eta^S_t - \eta^B(t,T_2)dW_t \tag{4}$$

for some constant b. By the isometry for stochastic integrals we get

$$\int_0^{T_2} |b\eta^S_t + (1-b)\eta^B(t,T_2) - \eta^B(t,T_1)|^2 dt \stackrel{!}{=} 0.$$

Finally, due to the continuity of the diffusion coefficients in t we have for $i = 1,\dots,d$

$$b\eta^S_{t,i} + (1-b)\eta^B_i(t,T_2) - \eta^B_i(t,T_1) \stackrel{!}{=} 0 \quad \forall\, t \in [0,T_2],$$

which proves the first part of the theorem. By (4) we see that $\ln B(T_2, T_1)$ is an affine transformation of $\ln S_{T_2}$, which proves the perfect correlation between the two. This completes the proof of the theorem.

Clearly condition (3) must be dismissed as unreasonable and unacceptable in any serious stochastic model of stock and bond markets. We therefore conclude that in general under the assumption of stochastic interest rates the decision whether or not to exercise the compound call depends on the realization of the pair of random variables S_{T_2} and $B(T_2, T_1)$. The set on which the compound call is exercised is given by $A = \{\omega \in \Omega | (S_{T_2}(\omega), B(T_2, T_1)(\omega)) \in \tilde{A}\}$, where

$$\tilde{A} = \{(S, B) \in \mathbb{R}_+^2 | \tilde{C}_{T_2}(S, B, K_1, T_1) > K_2\} \tag{5}$$

and not as in Geman et al. [GER95] by $A_1 = \{\omega \in \Omega | S_{T_2}(\omega) > S^*\}$ with S^* the supposedly unique value of S_{T_2} that equates the price of the underlying option at T_2 and the strike of the compound option.

2. Modified Geske–Formulas under Stochastic Interest Rates

A simple inspection of the formula for the call on a stock under stochastic interest rates shows that a Geske–type formula can still be obtained in this case if the second option is not written on the spot price of the underlying option but on the T_1 forward price of the latter since this is a strictly monotonically increasing function of the T_1 forward price of S alone. Given this modification, Geske's argument goes through unaltered even under stochastic interest rates and the exact formula can be obtained using the change of measure technique. Somewhat more interesting is the fact that a Geske–type formula also exists if the option underlying the compound call option is a futures–style option, i.e. if it is continuously marked to market. It is well known that this is equivalent to saying that the second option is written on a futures on the underlying option. We have

Proposition 2 **(i)** *Under Assumption 1.1 the price at $0 \le \tau \le T$ of a futures–style option on the stock S with maturity date T and strike price K is given by*

$$F(\tau, T)\mathcal{N}(d_1) - K\mathcal{N}(d_2),$$

where $\mathcal{N}$ is the cumulative standard normal distribution function and

$$
\begin{aligned}
F(\tau,T) &= \frac{S_\tau}{B(\tau,T)} \exp\left\{ -\int_\tau^T \eta_t^S \eta^B(t,T)dt + \int_\tau^T |\eta^B(t,T)|^2 dt \right\} \\
d_1 &= \frac{\ln F(\tau,T) - \ln K + \frac{1}{2}\int_\tau^T |\eta_t^S - \eta^B(t,T)|^2 ds}{\sqrt{\int_\tau^T |\eta_t^S - \eta^B(t,T)|^2 ds}} \\
d_2 &= d_1 - \sqrt{\int_\tau^T |\eta_t^S - \eta^B(t,T)|^2 ds}\,.
\end{aligned}
$$

(ii) *If the underlying of a compound call option with strike K_2 maturing at T_2 is a futures–style call option on S with strike K_1 maturing at $T_1 > T_2$, under Assumption 1.1 the price at time $0 \le t \le T_2$ of the compound call is given by*

$$X_t \mathcal{N}_2(d_1,d_2,\rho) - K_1 B(t,T_2)\mathcal{N}_2(e_1,e_2,\rho) - K_2 B(t,T_2)\mathcal{N}(e_1) \quad \textit{where}$$

$$
\begin{aligned}
X_t &= \frac{S_t B(t,T_2)}{B(t,T_1)} \exp\left\{ -\frac{1}{2}\int_t^T |\eta_s^S|^2 - |\eta^B(s,T_1)|^2 + |\eta^B(s,T_2)|^2 ds + \frac{1}{2}\int_s^T |\nu_X(s,T_1,T_2)|^2 ds \right\} \\
\nu_X &= \eta_s^S + \eta^B(s,T_2) - \eta^B(s,T_1) \\
d_1 &= \frac{\ln F(t,T_1) - \ln F^*(K_2) + \frac{1}{2}\int_t^{T_2} |\eta_s^S - \eta^B(s,T_1)|^2 ds + \int_t^{T_2} (\eta_s^S - \eta^B(s,T_1))\eta^B(s,T_2) ds}{\sqrt{\int_t^{T_2} |\eta_s^S - \eta^B(s,T_1)|^2 ds}} \\
d_2 &= \frac{\ln F(t,T_1) - \ln K_1 + \frac{1}{2}\int_t^{T_1} |\eta_s^S - \eta^B(s,T_1)|^2 ds + \int_t^{T_2} (\eta_s^S - \eta^B(s,T_1))\eta^B(s,T_2) ds}{\sqrt{\int_t^{T_1} |\eta_s^S - \eta^B(s,T_1)|^2 ds}} \\
\rho &= \frac{\int_t^{T_2} |\eta_s^S - \eta^B(s,T_1)|^2 ds}{\sqrt{\int_t^{T_2} |\eta_s^S - \eta^B(s,T_1)|^2 ds \int_t^{T_1} |\eta_s^S - \eta^B(s,T_1)|^2 ds}} \\
e_1 &= d_1 - \sqrt{\int_t^{T_2} |\eta_s^S - \eta^B(s,T_1)|^2 ds} \\
e_2 &= d_2 - \sqrt{\int_t^{T_1} |\eta_s^S - \eta^B(s,T_1)|^2 ds}\,,
\end{aligned}
$$

where $\mathcal{N}_2$ is the bivariate cumulative standard normal distribution function with correlation coefficient ρ and

Proof: ad **(i)** As pointed out a futures–style option can be regarded as a futures on an ordinary European option. It is well known (see [CIR81]) that in a complete market futures prices are martingales under the risk neutral measure, which we have denoted by P. Hence, we have for the price at τ of the futures–style option

$$E_\tau^P\left[[S_T - K]^+\right] = F(\tau,T)E_\tau^P\left[dQ/dP\, 1_{\{F(\tau,T)dQ/dP > K\}}\right] - K E_\tau^P\left[1_{\{F(\tau,T)dQ/dP > K\}}\right],$$

where $dQ/dP := \exp\left\{-\frac{1}{2}\int_\tau^{T_2}|\eta_s^S - \eta^B(s,T_1)|^2 ds + \int_\tau^{T_2}\eta_s^S - \eta^B(s,T_1)dW_s\right\}$ is a Radon Nikodym derivative. Applying Girsanov's theorem gives the required result.

ad **(ii)** By absence of arbitrage we have

$$\tilde{C}_t^{co} = E_t^P\left[\beta_{t,T_2}^{-1}\left[E_{T_2}^P[[S_{T_1} - K_1]^+] - K_2\right]^+\right]$$

By (i) the price of a futures–style call at T_2 is strictly monotonically increasing in $F(T_2, T_1)$. Hence, there is exactly one $F^*(K_2)$ such that this price is equal to K_2. Therefore, we may express the price of a compound option in the following way:

$$\tilde{C}_t^{co} = E_t^P\left[\beta_{t,T_2}^{-1}\left[(S_{T_1} - K_1)\, 1_{\{F(T_2,T_1)>F^*(K_2)\}} 1_{\{S_{T_1}>K_1\}} - K_2 1_{\{F(T_2,T_1)>F^*(K_2)\}}\right]\right]$$

Now note that $\beta_{t,T_2}^{-1}S_{T_1} = X_t \exp\left\{-\frac{1}{2}\int_t^{T_1}|\nu_X(s,T_1,T_2)|^2 ds + \int_t^{T_1}\nu_X(s,T_1,T_2)dW_s\right\}$, since we have defined $\eta^B(s,T) \equiv 0$ for $s \geq T$. Hence the result follows from Girsanov's theorem and some tedious but straightforward calculations.

3. A Near Explicit Solution for Geske's original Problem under Stochastic Interest Rates

Much as in general it is not possible to obtain a closed valuation formula for compound options under stochastic interest rates the change of measure technique still facilitates the numerical valuation of a compound option and the derivation of the hedge portfolio. We have

Proposition 3 **(i)** *Under Assumption 1.1 the price at time $0 \leq t \leq T_2$ of a compound call as specified in equation (1) is given by*

$$\begin{aligned}\tilde{C}_t^{co} &= S_t E_t^{Q^S}\left[\mathcal{N}(d_1(T_2))1_A\right] - K_1 B(t,T_1) E^{Q^{T_1}}\left[\mathcal{N}(d_2(T_2))1_A\right]\\ &\quad - K_2 B(t,T_2) Q^{T_2}\left[A\right],\end{aligned}$$

where d_1, d_2 and $\tilde{C}$ are as specified in Proposition 1, A is the set on which the compound call is exercised (see section 1), and $\frac{dQ^S}{dP} = \frac{\beta_{0,T_2}^{-1}S_{T_2}}{S_0}$, $\frac{dQ^{T_i}}{dP} = \frac{\beta_{0,T_i}^{-1}B(T_i,T_i)}{B(0,T_i)}$.

(ii) *The hedge portfolio for the compound call as specified in (i) is given by*

$\delta_S(t)$	$=$	$E_t^{Q^S}\left[\mathcal{N}(d_1(T_2))1_A\right]$	*units of the stock*
$\delta_{B(.,T_1)}(t)$	$=$	$-K_1 E^{Q^{T_1}}\left[\mathcal{N}(d_2(T_2))1_A\right]$	*units of the bond maturing at* T_1
$\delta_{B(.,T_2)}(t)$	$=$	$-K_2 Q^{T_2}\left[A\right]$	*units of the bond maturing at* T_2.

REMARK: Obviously under each of the above probability measures $\ln S_{T_2}$ and $\ln B(T_2,T_1)$ are bivariate normally distributed. Thus the evaluation of the above formula boils down to integrating the arguments of the above expectations over the set $\tilde{A}$ as in (5) with respect to the respective bivariate normal distribution corresponding to Q^S, Q^{T_1} and Q^{T_2}. There are a number of numerical techniques to perform these integrations. For a more detailed discussion of this point see section 4 of [FS96]. Also notice that due to the change of measure technique the price of the option and the hedge portfolio can be derived in a single step and with equal precision.

PROOF

(i) By absence of arbitrage we have

$$\tilde{C}_t^{co} = E_t^P\left[\beta_{t,T_2}^{-1}\Big(S_{T_2}\mathcal{N}(d_1(T_2)) - K_1 B(T_2,T_1)\mathcal{N}(d_2(T_2)) - K_2 B(T_2,T_2)\Big)1_A\right].$$

Using the change of measure technique to introduce Q^S and Q^{T_i} as defined in the proposition the first statement immediately follows.

(ii) We need to compute the martingale part of the price process of the compound option. In order to apply Ito's formula as a first step we have to compute the derivatives of the compound call with respect to S_t, $B(t,T_1)$ and $B(t,T_2)$. Consider the example of $\partial\tilde{C}_t^{co}/\partial S_t$. Exchanging integration and differentiation as in [EKMV92b] we obtain

$$\begin{aligned}
\frac{\partial}{\partial S_t}\tilde{C}_t^{co} &= E_t\left[\frac{\partial}{\partial S_t}\left[\beta_{t,T_2}^{-1}\cdot(\tilde{C}_{T_2} - B(T_2,T_2)K)\right]^+\right] \\
&= E_t\left[1_A\cdot\frac{\partial}{\partial S_t}\left[\beta_{t,T_2}^{-1}\cdot(\tilde{C}_{T_2} - B(T_2,T_2)K)\right]\right] \\
&= E_t\left[1_A\cdot\mathcal{N}(d_1(T_2))\beta_{t,T_2}^{-1}\cdot\frac{S_{T_2}}{S_t}\right] \\
&= E_t^{Q^S}\left[1_A\cdot\mathcal{N}(d_1(T_2))\right]
\end{aligned}$$

Similarly we get

$$\frac{\partial}{\partial B(t,T_1)}\tilde{C}_t^{co} = -K_1 E^{Q^{T_1}}\left[\mathcal{N}(d_2(T_2))1_A\right] \text{ and } \frac{\partial}{\partial B(t,T_2)}\tilde{C}_t^{co} = -K_2 Q^{T_2}\left[A\right]$$

Hence by Ito's formula the martingale part $[\tilde{C}_t^{co}]^M$ of the price process of the compound option is given by

$$\begin{aligned} d[\tilde{C}_t^{co}]^M &= E_t^{Q^S}\left[\mathcal{N}(d_1(T_2)1_A\right] d[S_t]^M - K_1 E_t^{Q^S}\left[\mathcal{N}(d_1(T_2)1_A\right] d[B(t,T_1)]^M \\ &\quad -K_2 Q^{T_2}\left[A\right] d[B(t,T_2)]^M. \end{aligned}$$

Using the self-financing condition for the value process of the hedge portfolio we see that this is the same as the martingale part of the value process of the proposed portfolio. Now observe that the price process of the compound call is linear homogeneous in S, $B(\cdot,T_1)$ and $B(\cdot,T_2)$. Hence the value of the proposed hedge portfolio equals the price of the compound call for all $0 \leq t \leq T_2$ which completes the proof. For a more detailed version of these arguments in a similar context see the proof of Theorem 4.1 in [FS96].

4. Conclusion

In this note we have explored some of the difficulties that arise when one tries to generalize Geske's formula for compound options to the case of stochastic interest rates. We have pointed out some special cases where a closed formula still exists and have sketched an efficient numerical procedure for calculating the option price and the hedge portfolio when closed formulas are no longer available.

A general feature that emerges from this discussion is that the change of measure technique proposed by Geman et al. [GER95] is a powerful tool for dealing with option pricing models that incorporate interest rate risk. In fact, as long as non path dependent European options are considered due to this technique deriving valuation formulas in a framework of stochastic interest rates is no more difficult than deriving them under deterministic interest rates. However, as soon as path dependent options are considered difficulties may arise that cannot be solved by the change of measure technique. Hence, situations may occur in which a closed valuation formula that exists under deterministic interest rates cannot be generalized to a framework with stochastic interest rates. A case in point is the compound call considered in this note. But even in this case the change of measure technique is helpful in designing efficient numerical valuation procedures.

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