NONLINEAR BLACK–SCHOLES EQUATIONS IN FINANCE: ASSOCIATED CONTROL PROBLEMS AND PROPERTIES OF SOLUTIONS[*]

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Abstract. We study properties of solutions to fully nonlinear versions of the standard Black–Scholes partial differential equation. These equations have been introduced in financial mathematics in order to deal with illiquid markets or with stochastic volatility. We show that typical nonlinear Black–Scholes equations can be viewed as dynamic programming equation of an associated control problem. We establish existence and comparison results and show that the equation induces a convex risk measure on the set of all continuous terminal value claims. Moreover, we study the asymptotic behavior of solutions as market frictions get "large." Finally, the pricing of individual contracts relative to a book of derivatives is discussed.

1. Introduction. While the standard Black–Scholes model was the single most important step in the development of modern derivative asset analysis, the underlying assumptions of constant volatility and of a perfectly liquid market are clearly at odds with reality. As a consequence a number of approaches for dealing with the pricing and the hedging of derivatives in markets with limited liquidity or with stochastic volatility have been developed. Often prices and hedging strategies in these models are described by fully nonlinear versions of the standard parabolic Black–Scholes partial differential equation (PDE); see, for instance, [13], [7], or [2]. A brief overview, including further references, is given in section 2. It turns out that despite substantial differences in the underlying financial framework, these nonlinear Black–Scholes equations have a very similar structure, making them a useful tool for measuring the risk management cost for a (book of) derivatives in illiquid markets or in markets with stochastic volatility.

In this paper we are interested in properties of solutions to typical nonlinear Black–Scholes equations. Our starting point is the observation that after a minor modification the equations can be viewed as Hamilton–Jacobi–Bellmann (HJB) equation of an associated stochastic control problem. Moreover, this control problem has a natural economic interpretation. The HJB equation is studied in detail in section 3. We establish existence and comparison results for classical and viscosity solutions. Moreover, we show that the equation induces a convex risk measure on the set of all continuous terminal value claims (derivatives with payoff $h(S_T)$), and we use the control problem associated with the equation to give a dual representation of this risk

measure in the sense of [11]. Section 4 is concerned with asymptotic properties of solutions to the nonlinear Black–Scholes equations: it is shown that for large market frictions the solution converges to the concave envelope of the payoff $h(S_T)$. Clearly, both properties are fully in line with economic intuition. The latter half of the paper is devoted to specific applications. In section 5 we discuss the application of our general results on nonlinear Black–Scholes equations to the illiquid market models of [13] and of [6]. In section 6 we finally explain how the control problem associated with the modified nonlinear Black–Scholes equation can be used to determine prices for individual contracts in a book of derivatives in a way that is consistent with the contribution of each contract to the risk management cost of the overall position.

We are not aware of similar results in the literature. On the technical side our work is related to the papers from the dynamic programming approach to superreplication under stochastic volatility or liquidity cost, most notably [9] and [7]. Further related references are given in the body of the paper.

2. Nonlinear Black–Scholes equations in derivative asset analysis. In order to put the subsequent analysis into context we briefly discuss a number of financial models leading to nonlinear Black–Scholes equations for the risk management cost associated with path independent derivative securities. We begin with two models for pricing and hedging of derivatives in the presence of liquidity risk, followed by the uncertain volatility model of [2]. In all models there will be two assets, a risk-free money-market account B, which is perfectly liquid, and a risky and illiquid asset S (the stock). We work directly with discounted quantities; hence $B_t \equiv 1$, S represents the forward price of the stock, and interest rates can be taken equal to zero. Throughout we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting a Brownian motion W.

Models for illiquid markets can be grouped into two classes. On the one hand, there are models in which the impact of trading on the stock price is purely temporary, reflecting mainly a widening of the bid-offer spread in reaction to the proposed trade. On the other hand, there are models in which the price impact is permanent. In this class one attempts to model the effect of the additional supply or demand created by hedging activities on the equilibrium price of the stock.

2.1. Illiquid market models with temporary price impact. The predominant model in this class has been put forward by [6]; see also [5]. For our purposes it is enough to concentrate on a special case of the CJP-framework, the so-called extended Black–Scholes economy. In this economy there is a fundamental stock price process S^0, which follows geometric Brownian motion with volatility $\sigma > 0$. The *transaction price* to be paid at time t for trading α shares is

$$\bar{S}_t(\alpha) = e^{\rho\alpha} S_t^0, \quad \alpha \in \mathbb{R}, \ \rho \geq 0. \tag{1}$$

Note that in the model (1) the trader has to pay a bid-ask spread, whose size depends on the parameter ρ and on the amount α which is traded. The parameter ρ models the liquidity of the market: for $\rho = 0$ the market is perfectly liquid, whereas for ρ large a trade has a substantial impact on the transaction price. Empirical evidence from [5] shows that for the stock of major U.S. corporations ρ is small (of the order of 10^{-4}) but significantly different from zero.

As shown in [6], under the model (1) the liquidity cost of implementing a continuous stock trading strategy ϕ is proportional to the quadratic variation $[\phi]_t$ of the strategy. More precisely, consider a self-financing trading strategy with stock position

ϕ_t, bond position η_t, and value $V_t = \phi_t S_t^0 + \eta_t$. In line with the standard Black–Scholes model let $\phi_t = \varphi(t, S_t^0)$ for a smooth function φ. Theorem A3 of [6] then yields the following dynamics of V_t:

$$(2) \qquad dV_t = \varphi(t, S_t^0) dS_t^0 - \rho S_t^0 d[\phi]_t = \varphi(t, S_t^0) dS_t^0 - \rho S_t^0 \varphi_S^2(t, S_t^0) \sigma^2 (S_t^0)^2 dt,$$

and the term $\rho S_t^0 \varphi_S^2(t, S_t^0) \sigma^2 (S_t^0)^2 dt$ can be viewed as additional liquidity cost. We remark that V_t is the so-called paper value of the position; under (1) the liquidation value of the strategy (the amount of money the large trader receives if he actually liquidates his stock position) will be lower than V_t. Following [7] we concentrate on the paper-value concept, liquidation values are discussed, for instance, in [3].

Suppose now that u and φ are smooth functions and that $u(t, S_t^0)$ gives the value of a self-financing trading strategy with stock position $\varphi(t, S_t^0)$. According to the Itô formula, $u(t, S_t^0)$ has dynamics

$$du(t, S_t^0) = u_S(t, S_t^0) dS_t^0 + \left(u_t(t, S_t^0) + \frac{1}{2} \sigma^2 (S_t^0)^2 u_{SS}(t, S_t^0) \right) dt.$$

Comparing this with (2) it is immediate that u must satisfy the equation $u_t + \frac{1}{2} \sigma^2 S^2 u_{SS} + \rho S^3 \sigma^2 \varphi_S^2 = 0$ and that $\varphi = u_S$. Hence $\varphi_S = u_{SS}$, and we obtain the following nonlinear PDE for u:

$$(3) \qquad u_t + \frac{1}{2} S^2 v^{\mathrm{CJP}}(S, u_{SS}) = 0 \text{ with } v^{\mathrm{CJP}}(S, q) = \sigma^2 q(1 + 2\rho S q).$$

Note that for $\rho = 0$, (3) reduces to the standard linear Black–Scholes PDE.

Consider now a terminal value claim with payoff $h(S_T)$. It follows that the value of a self-financing replicating strategy for this claim is given by the solution u of the PDE (3) with boundary condition $u(T, S) = h(S)$ (provided that this equation admits a solution). The corresponding strategy is then given by $\phi_t = u_S(t, S_t)$. In a recent paper [7] it was shown that u is indeed the superreplication price of h provided that h is convex. In more general situations the superreplication price of h can be described by the *parabolic envelope* of (3). This is a PDE of the form (3) but with $v^{\mathrm{CJP}}(S, \cdot)$ replaced by the largest increasing minorant $\widehat{v^{\mathrm{CJP}}}(S, \cdot)$ of $v^{\mathrm{CJP}}(S, \cdot)$. This is discussed in detail in section 5.2.

2.2. Equilibrium or reaction function models. Here the starting point of the analysis is a smooth *reaction function* Ψ that gives the equilibrium stock price S_t as a function of some fundamental value F_t and of the stock position ϕ_t of the large trader at time t; i.e., one has the relation $S_t = \Psi(F_t, \phi_t)$. The function Ψ can be seen as a reduced form representation of an economic equilibrium model such as the models proposed by [15], [21], or [22]. Variants of the reaction function approach are also used in [17], [12], and [3]. For concreteness we concentrate on the model from [21]. Here

$$(4) \qquad \Psi(f, \phi) = f \exp(\rho \phi), \quad \rho \geq 0 \text{ a liquidity parameter,}$$

and the process F follows a geometric Brownian motion with volatility σ. As before we assume that the strategy of the large trader is of the form $\phi_t = \varphi(t, S_t)$ for a smooth function φ. Using Itô's formula, equation (4), and the fact that F is geometric Brownian motion we get that

$$(5) \quad dS_t = \Psi_f(F_t, \phi_t) dF_t + \Psi_\alpha(F_t, \phi_t) d\phi_t + \cdots + dt = \sigma S_t dW_t + \rho S_t d\phi_t + \cdots + dt;$$

the precise form of the dt-terms is irrelevant. In the model (4) it is assumed that the variation of the large trader's trading strategy is small relative to the market in the sense that

$$(6) \qquad \qquad \left(1 - \rho S_t \varphi_S(t, S_t)\right) > 0 \quad \text{a.s.}$$

Plugging the relation $d\phi_t = \varphi_S(t, S_t)dS_t + \cdots + dt$ into (5), rearranging terms, and integrating $\left(1 - \rho S_t \varphi_S(t, S_t)\right)^{-1}$ over both sides then gives the following dynamics of S:

$$(7) \qquad dS_t = \sigma_{[\varphi]}(t, S)dW_t + \cdots + dt \ \text{ with volatility } \ \sigma_{[\varphi]}(t, S) := \frac{\sigma}{1 - \rho S \varphi_S(t, S)}.$$

Consider now a self-financing trading strategy with value $V_t = u(t, S_t)$ and stock position $\varphi(t, S_t)$. By standard arguments we get that u satisfies the equation $u_t + \frac{1}{2}\sigma^2_{[\varphi]}(t, S)S^2 u_{SS} = 0$ and that $\varphi = u_S$. Using the definition of $\sigma_{[\varphi]}$ we obtain the fully nonlinear PDE

$$(8) \qquad \qquad u_t + \frac{1}{2}\frac{\sigma^2}{(1 - \rho S u_{SS})^2}S^2 u_{SS} = 0.$$

In order to derive a hedging strategy for a claim with payoff h we add the terminal condition $u(T, \cdot) \equiv h$. As before, $u(t, S_t)$ gives the cost of implementing the strategy, and $u_S(t, S_t)$ gives the position in the risky asset, provided that the candidate hedge $u_S(t, S_t)$ satisfies the condition (6).

Comments. 1. The related papers [13] and [20] specify directly the stock price dynamics resulting from a given strategy of the large trader: if he uses a semimartingale trading strategy ϕ, the stock price has differential $dS_t = \sigma S_t dW_t + \rho S_t d\phi_t$, as in (5). This again leads to the nonlinear Black–Scholes PDE (8); the derivation is identical to the one given here.

2. Since ρ is usually considered to be a small parameter, it is natural to replace the "coefficient" of u_{SS} in (8) by the first order Taylor approximation around $\rho = 0$, given by

$$\frac{\sigma^2 S^2}{(1 - \rho S u_{SS})^2} \approx \sigma^2 S^2 \left(1 + 2\rho S u_{SS}\right) + o(\rho).$$

Substituting this relation into (8) immediately leads to the PDE (3). This shows that the nonlinear PDEs arising in the CJP-model and in the reaction-function setting are closely related, despite the differences in the underlying economic framework.

2.3. Uncertain volatility. Finally, we turn to the uncertain volatility model of [2]. Other than in the previous two model classes, here the option hedger is a small investor. It is assumed that the stock price follows a diffusion process of the form $dS_t = \sigma_t S_t dW_t$. The dynamics of the volatility σ_t is not specified; [2] merely assume that there are bounds $0 < \underline{\sigma} < \overline{\sigma} < \infty$ such that

$$(9) \qquad \qquad \underline{\sigma} \leq \sigma_t \leq \overline{\sigma} \quad \text{ for all } 0 \leq t \leq T.$$

Consider as before an option with payoff $h(S_T)$. Suppose that the function u solves the following nonlinear PDE (the so-called Barenblatt equation):

$$(10) \qquad \qquad u_t + \frac{1}{2}S^2 u_{SS}\left(\underline{\sigma}^2\mathbf{1}_{\{u_{SS}<0\}} + \overline{\sigma}^2\mathbf{1}_{\{u_{SS}\geq 0\}}\right) = 0$$

with terminal condition $u(T, S) = h(S)$. Then it is shown in [2] that the strategy with initial value $V_0 = u(0, S_0)$ and stock position $\phi_t = u_S(t, S_t)$ is a superreplication strategy for the option for all volatility processes that satisfy the volatility bounds (9). The intuition for this result is straightforward: the PDE (10) corresponds to the option price in a worst-case volatility scenario, where $\sigma_t = \overline{\sigma}$ whenever the superreplication price $u(t, S)$ is locally convex in S and where $\sigma_t = \underline{\sigma}$ whenever u is locally concave.

3. Properties of nonlinear Black–Scholes equations.

3.1. Two boundary value problems.

In this section we formally introduce the nonlinear pricing PDEs which will be studied in what follows. In order to avoid technical difficulties related to the analysis of PDEs on unbounded domains we study a terminal-boundary value problem.[1] Consider for $0 < \underline{S} < \overline{S} < \infty$ the set $Q := [0, T) \times (\underline{S}, \overline{S})$ with closure $\overline{Q}$. Following [10], we define the parabolic boundary of Q by $\partial^* Q := \big([0, T] \times \{\underline{S}, \overline{S}\}\big) \cup \big(\{T\} \times (\underline{S}, \overline{S})\big)$. We consider a terminal payoff of the form $h \colon [\underline{S}, \overline{S}] \to \mathbb{R}$ and extend h to a function $\tilde{h}$ on $\partial^* Q$ by setting

$$\tilde{h}(t, \underline{S}) := h(\underline{S}) \quad \text{and} \quad \tilde{h}(t, \overline{S}) = h(\overline{S}) \text{ for } 0 \leq t \leq T. \tag{11}$$

Note that $\tilde{h}(t, \underline{S})$ and $\tilde{h}(t, \overline{S})$ can be viewed as a rebate in the case when the asset price exits the layer $(\underline{S}, \overline{S})$ before maturity T. In applications it is implicitly understood that $\underline{S}$ is small and that $\overline{S}$ is large relative to the current asset price; hence the precise form of $\tilde{h}$ is irrelevant for the economic interpretation of our results.

The original problem. The starting point of our analysis is the following generic terminal-boundary value problem, also labeled *original* problem:

$$u_t + \frac{1}{2} S^2 v(S, u_{SS}) = 0 \,, \quad (t, S) \in Q, \tag{12}$$

$$u = \tilde{h} \,, \quad (t, S) \in \partial^* Q \,. \tag{13}$$

We make the following assumptions on the data of the problem.

(A1) The payoff $h \colon [\underline{S}, \overline{S}] \to \mathbb{R}$ is continuous, and $\tilde{h}$ is constructed from h as in (11). The function $v \colon [\underline{S}, \overline{S}] \times \mathbb{R} \to \mathbb{R}$ is continuous on the set $\mathrm{dom}(v) := \{(S, q) \in [\underline{S}, \overline{S}] \times \mathbb{R} \colon v(S, q) < \infty\}$.

(A2) For fixed $S \in [\underline{S}, \overline{S}]$ the mapping $v(S, \cdot) \colon q \mapsto v(S, q)$ is convex and lower semicontinuous. Moreover, $v(S, 0) = 0$, and there is a constant $\lambda^0 > 0$ with $v_q^-(S, 0) \leq \lambda^0 \leq v_q^+(S, 0)$ for all $S \in [\underline{S}, \overline{S}]$, where v_q^- and v_q^+ denote the left and right derivative of the convex function $v(S, \cdot)$.

The convexity of $v(S, \cdot)$ will be crucial for our analysis. Assumptions A1 and A2 are satisfied for the nonlinear PDEs introduced in the previous section. In the PDE (3) from the CJP-model we have

$$v(S, q) = v^{\mathrm{CJP}}(S, q; \rho, \sigma) := \sigma^2 q(1 + 2\rho S q) \text{ for } (S, q) \in [\underline{S}, \overline{S}] \times \mathbb{R}; \tag{14}$$

in the PDE (8) from the models of [21] and [13] we have

$$v(S, q) = v^{\mathrm{reac}}(S, q; \rho, \sigma) := \begin{cases} \sigma^2 \frac{q}{(1 - \rho S q)^2} & \text{for } 1 - \rho S q > 0, \\ \infty & \text{otherwise,} \end{cases} \tag{15}$$

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[1]We are confident that under strong growth conditions most results in this paper can be extended to the case of a stock price in the domain $(0, \infty)$. However, this leads to considerable technical difficulties without yielding much extra economic insight, so we refrain from such an analysis.

so that $\mathrm{dom}\, v = \{(S, q) \in [\underline{S}, \overline{S}] \times \mathbb{R} \colon 1 - \rho S q > 0\}$; in the PDE (10) corresponding to the uncertain volatility model of [2] we have

$$(16) \quad v(S, q) = v^{\mathrm{uv}}(S, q; \underline{\sigma}, \overline{\sigma}) := q\left(\underline{\sigma}^2 \mathbf{1}_{\{q<0\}} + \overline{\sigma}^2 \mathbf{1}_{\{q\geq 0\}}\right) \text{ for } (S, q) \in [\underline{S}, \overline{S}] \times \mathbb{R}.$$

Whenever possible the parameters $\rho, \sigma, \underline{\sigma}, \overline{\sigma}$ will be omitted from the notation.

   The modified problem. In what follows we will often work with a modified version of (12). For fixed $S \in [\underline{S}, \overline{S}]$, denote by

$$(17) \qquad\qquad v^*(S, \cdot)\colon \mathbb{R} \to [0, \infty], \quad \lambda \mapsto \sup\{\lambda q - v(S, q)\colon q \in \mathbb{R}\}$$

the conjugate function of $v(S, \cdot)$. As $v(S, \cdot)$ is convex and lower semicontinuous by A2, the duality theorem for conjugate functions yields

$$(18) \qquad\qquad v(S, q) = \sup\left\{\lambda q - v^*(S, \lambda)\colon \lambda \in \mathbb{R}\right\}.$$

Consider constants $0 \leq \underline{v} \leq \lambda^0 \leq \overline{v} < \infty$ (typically $\underline{v}$ small and $\overline{v}$ large), and define in analogy with (18)

$$(19) \qquad\qquad \widetilde{v}(S, q) = \sup\left\{\lambda q - v^*(S, \lambda)\colon \lambda \in [\underline{v}, \overline{v}]\right\}.$$

Note that $\widetilde{v}$ is in general better behaved than v: we have $\mathrm{dom}\, \widetilde{v} = [\underline{S}, \overline{S}] \times \mathbb{R}$, and the mapping $q \mapsto \widetilde{v}(S, q)$ is increasing with $\underline{v} \leq \widetilde{v}_q(S, q) \leq \overline{v}$; see section 5 for details. The PDE

$$(20) \qquad\qquad u_t + \frac{1}{2}S^2\widetilde{v}(S, u_{SS}) = 0, \quad (t, S) \in Q,$$

will be called the *modified* nonlinear Black–Scholes equation. From a mathematical viewpoint this equation has very desirable properties: the modified equation (20) is parabolic—which is in general not true for the original problem—and it has a natural interpretation as a dynamic programming equation.

   For the function v^{uv} from the uncertain volatility model, v is equal to $\widetilde{v}$ provided that $\underline{v} = \underline{\sigma}^2$ and $\overline{v} = \overline{\sigma}^2$, so that the original and the modified problem coincide. For the illiquid market models, on the other hand, the equality $v(S, q) = \widetilde{v}(S, q)$ holds only if $|q|$ is not too large relative to the liquidity parameter ρ (see Figure 1), so that the two problems are in general different. This calls for a justification of the modified Black–Scholes equation from a financial point of view, and we have the following arguments to offer: To begin with, it will be shown in section 5 that by and large the results derived for the modified equation (20) apply to the illiquid market models as well. In particular, for a smooth payoff, solutions of the modified boundary problem solve also the original problem. Moreover, the superreplication price in the CJP-model is the limit of an increasing sequence of solutions to the modified equation with $\overline{v}_n \to \infty$. Finally, the modified equation—and its limit as $\overline{v}_n \to \infty$—induces a convex risk measure on the space of all terminal value claims. This is a very desirable feature of any methodology that attempts to measure the risk management cost of a book of derivatives in an incomplete or illiquid market.

3.2. Interpretation as an HJB equation. In this subsection we show that the modified nonlinear Black–Scholes equation (20) can be viewed as a formal HJB equation of an associated stochastic control problem. This observation will be fundamental for the subsequent analysis.
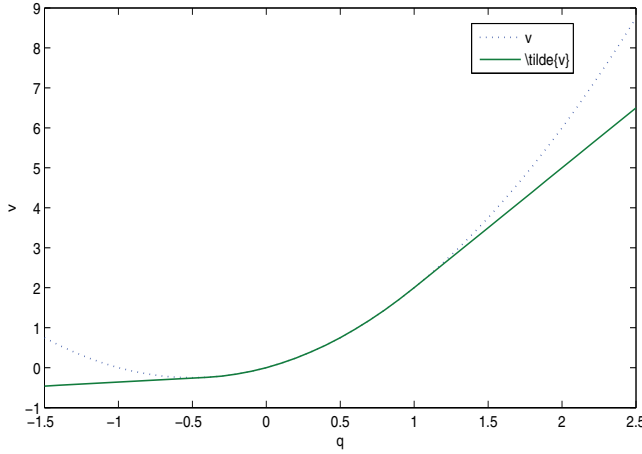
FIG. 1. *A graph of the function v^{CJP} and of the modified function $\widetilde{v}$ for $\underline{v} = 0.2$ and $\overline{v} = 3$.*

Denote by $\mathcal{U}^{[\underline{v},\overline{v}]}$ the set of all progressively measurable processes $\Lambda = (\lambda_t)_{0 \leq t \leq T}$ with values in $[\underline{v}, \overline{v}]$. Assume that for a given control $\Lambda \in \mathcal{U}^{[\underline{v},\overline{v}]}$ the state process S has dynamics

$$(21) \qquad dS_t = \sqrt{\lambda_t} S_t dW_t, \quad S_0 = S \in (\underline{S}, \overline{S}).$$

Define the stopping time $\tau = \inf\{t \geq 0 \colon (t, S_t) \notin \overline{Q}\}$ (note that $\tau \leq T$ by definition), and for $(t, S) \in Q$ and $\Lambda \in \mathcal{U}^{[\underline{v},\overline{v}]}$ let

$$(22) \qquad J(t, S, \Lambda) := E_{t,S}\left(\int_t^\tau -\frac{1}{2} S_s^2 v^*(S_s, \lambda_s)\, ds + \tilde{h}(\tau, S_\tau)\right),$$

$$(23) \qquad J^*(t, S) := \sup\left\{J(t, S, \Lambda) \colon \Lambda \in \mathcal{U}^{[\underline{v},\overline{v}]}\right\}.$$

The HJB equation associated with the control problem (21), (22) is

$$(24)$$
$$u_t + \sup\left\{\frac{1}{2}S^2\lambda u_{SS} - \frac{1}{2}S^2 v^*(S, \lambda) \colon \lambda \in [\underline{v}, \overline{v}]\right\} = 0 \text{ for } (t, S) \in Q, \quad u = \tilde{h} \text{ on } \partial^* Q.$$

By definition of $\widetilde{v}$ in (19), this equation is identical to the modified nonlinear Black–Scholes equation (20).

The control problem (21), (22) admits the following interpretation: The controller, nature, say, chooses the path $(\sqrt{\lambda_t})_{0 \leq t \leq T}$ of the stock price volatility in order to maximize the expected value $E(\tilde{h}(\tau, S_\tau))$ of the payoff; in doing so she faces an instantaneous control cost of size $\frac{1}{2}S_t^2 v^*(S_t, \lambda_t)$. Note that the properties of $v^*(S, \cdot)$ imply that the control cost is nonnegative and convex in λ_t. In the uncertain volatility model the control cost $(v^{\mathrm{uv}})^*$ vanishes for $\lambda \in [\underline{\sigma}^2, \overline{\sigma}^2]$ so that the optimal strategy of nature is to switch between the squared volatility bounds $\underline{\sigma}^2$ and $\overline{\sigma}^2$. In the illiquid market models, on the other hand, $v^*(S, \lambda) > 0$ for $\lambda \neq 0$, and inner values $\lambda_t \in (\underline{v}, \overline{v})$ may be optimal.

3.3. Existence and comparison results. The dynamic programming, or HJB, equation (24) has been studied extensively in the literature; see, for instance, [10].

Hence we may use known results for dynamic programming equations to give existence, uniqueness, and comparison results for the modified nonlinear Black–Scholes equation (20).

Classical existence. Here we have the following result.

THEOREM 3.1. *Let assumptions* A1 *and* A2 *hold. Suppose that $\underline{v} > 0$, that v^* is smooth on $[\underline{S}, \overline{S}] \times [\underline{v}, \overline{v}]$ and that the payoff h can be extended to a $\mathcal{C}^3$-function on $(0, \infty)$. Then the boundary value problem* (20), (13) *admits a solution $u \in \mathcal{C}^{1,2}(Q) \cap \mathcal{C}(\overline{Q})$. Moreover, $\|u_{SS}\|_\infty$ is bounded by a constant which depends only on $\underline{v}$, $\overline{v}$, $\|v^*\|_\infty$ and on the Hölder norm of h_S and h_{SS}.*

Proof. The result follows from Theorem 6.4.1(b) and Example 6.1.8 of [19]; see also Theorem IV.4.1 of [10]. It is straightforward to check that the assumptions of that theorem are satisfied. Note in particular that (20) is uniformly parabolic, as

$$\frac{1}{2}S^2\lambda \geq \frac{1}{2}\underline{S}^2\underline{v} > 0, \quad (S, \lambda) \in [\underline{S}, \overline{S}] \times [\underline{v}, \overline{v}],$$

and that the control set $[\underline{v}, \overline{v}]$ is compact. □

Uniqueness is discussed in the context of viscosity solutions; see Theorem 3.3.

Viscosity solutions. Alternatively, we may consider viscosity solutions of the modified boundary value problem. We refer the reader to [8] or to [10] for background information regarding this solution concept. We remark that in the literature one typically considers (20) in the form $-u_t - \widetilde{v}(S, u_{SS}) = 0$. In particular, sign conventions in the definition of sub- and supersolutions correspond to this latter form.

Our first result is concerned with the characterization of the value function J^*.

PROPOSITION 3.2. *Under* (A1) *and* (A2) *J^* is a viscosity solution of the modified boundary value problem* (20), (13).

Note that Proposition 3.2 requires weaker regularity assumptions on the data of the problem than Theorem 3.1: we may allow for $\underline{v} = 0$, and the payoff function $h(T, \cdot)$ is merely assumed to be continuous instead of $\mathcal{C}^3$.

Proof. According to [10, Corollary V.3.1], J^* is a viscosity solution of the HJB equation (20) if $J^* \in \mathcal{C}(\overline{Q})$ and if J^* moreover satisfies a suitable dynamic programming principle. Sufficient conditions for this are given in [10, Theorem V.2.1], and we now check the applicability of this result. Conditions (IV.6.1) and (IV.6.3) from that theorem are obviously satisfied. An inspection of the proof of [10, Theorem V.2.1] shows that Condition (V.2.8) is needed only to ensure that there is some Markov control in $\mathcal{U}^{[\underline{v}, \overline{v}]}$ such that for all $0 < s \leq T - t$,

$$P_{t,\underline{S}}\left(\int_t^s (\underline{S} - S_r)^+ \, dr > 0\right) = P_{t,\overline{S}}\left(\int_t^s (S_r - \overline{S})^+ \, dr > 0\right) = 1,$$

where S is the state process from (21). Taking $\Lambda \equiv \lambda^0$ so that S follows a geometric Brownian motion, these conditions can be easily verified directly.

It remains to verify Condition (V.2.11). For this we need to find a smooth and bounded subsolution g of (20) on $[0, T] \times (0, \infty)$ such that $g(t, \underline{S}) = h(\underline{S})$, $g(t, \overline{S}) = h(\overline{S})$, and $g(T, S) \leq h(S)$ on $(\underline{S}, \overline{S})$. In order to construct g we choose $0 < \underline{M} < \underline{S} < \overline{S} < \overline{M} < \infty$ and a smooth and bounded function $\psi: (0, \infty) \to \mathbb{R}$, which is convex on $[\underline{M}, \overline{M}]$ and which moreover satisfies $\psi \leq h$, $\psi(\underline{S}) = h(\underline{S})$, and $\psi(\overline{S}) = h(\overline{S})$. Moreover, we extend the dynamic programming equation (24) (which is equivalent to (20)) to an equation on $[0, T] \times (0, \infty)$ as follows. Choose a smooth function $\nu: (0, \infty) \to [0, \infty)$ such that $\nu(S) = 1$ for $S \in [\underline{S}, \overline{S}]$ and $\nu(S) = 0$ for $S \leq \underline{M}$ or

$S \geq \overline{M}$, and consider the equation

$$u_t + \sup\left\{\frac{1}{2}(\nu(S)S)^2 \lambda u_{SS} - \frac{1}{2}(\nu(S)S)^2 v^*(S,\lambda) \colon \lambda \in [\underline{v}, \overline{v}]\right\} = 0, \ \ (t,S) \in [0,T) \times (0,\infty).$$

Then the constant function $\kappa(t,S) = \psi(S)$ is obviously a subsolution of this equation with $\kappa(T,S) \leq h(S)$ on $[\underline{S}, \overline{S}]$, and Condition (V.2.11) holds. $\quad\square$

Comparison principle. Next we derive a comparison principle for viscosity solutions of (20). We need the following assumption.

(A3) The functions $v^*(S,\lambda)$ and $v_S^*(S,\lambda)$ are continuous on $[\underline{S}, \overline{S}] \times [\underline{v}, \overline{v}]$.

The following result is an immediate consequence of [10, Lemma V.7.1 and Theorem V.8.1] applied to the HJB equation (20).

THEOREM 3.3. *Suppose that assumptions* A1, A2, *and* A3 *hold. Let* $u_1 \in \mathcal{C}(\overline{Q})$ *be a viscosity subsolution of* (20) *and* $u_2 \in \mathcal{C}(\overline{Q})$ *be a viscosity supersolution of* (20). *Then*

$$\sup\left\{u_1(t,S) - u_2(t,S) \colon (t,S) \in \overline{Q}\right\} = \sup\left\{u_1(t,S) - u_2(t,S) \colon (t,S) \in \partial^* Q\right\}.$$

It follows that if $u_1 \equiv u_2$ on $\partial^* Q$, then $u_1 \leq u_2$ in Q. Since a viscosity solution of (20) is both a sub- and a supersolution, under (A1)–(A3) uniqueness holds for viscosity solutions of the boundary value problem (20), (13). In particular, under (A1)–(A3) the value function J^* is the unique viscosity solution of the problem (20), (13).

Example 3.4. The comparison principle can be used to establish bounds on solutions of the modified problem (20), (13). Since under (A2) we have $v^*(S,\lambda^0) = 0 \leq v^*(S,\lambda)$ for all $(S,\lambda) \in [\underline{S}, \overline{S}] \times [\underline{v}, \overline{v}]$, the following inequalities hold:

$$(25) \qquad v^{\min}(S,q) := \lambda_0 q \leq \widetilde{v}(S,q) \leq \underline{v}q 1_{\{q<0\}} + \overline{v}q 1_{\{q\geq0\}} =: v^{\max}(S,q).$$

Note, moreover, that $v^{\min}$ and $v^{\max}$ are both of the form (19) with $(v^{\min})^*(S,\lambda) = 0$ for $\lambda = \lambda^0$ and $(v^{\min})^*(S,\lambda) = \infty$ otherwise, and with $(v^{\max})^*(S,\lambda) \equiv 0$. In economic terms $v^{\min}$ corresponds to the standard Black–Scholes equation with volatility $\sqrt{\lambda_0}$, and $v^{\max}$ corresponds to the Barenblatt equation (10) from the uncertain volatility model. Consider now a (viscosity) subsolution $u^{\min}$ and a supersolution $u^{\max}$ of the equations

$$u_t^{\min} + \frac{1}{2}S^2\lambda^0 u_{SS}^{\min} = 0 \ \text{ and } \ u_t^{\max} + \frac{1}{2}S^2 v^{\max}(S, u_{SS}^{\max}) = 0,$$

both with boundary condition (13). Let $\varphi \in \mathcal{C}^{1,2}$ be a test function with $(\varphi_t, \varphi_s, \varphi_{ss}) \in \mathcal{P}^{2,+}(u^{\min}(t,s))$ (the superjet of $u^{\min}$ in the point (t,S)). Using the definition of subsolutions and the left part of the inequality (25) we get

$$0 \leq \varphi_t + \frac{1}{2}S^2\lambda^0 \varphi_{SS} \leq \varphi_t + \frac{1}{2}S^2 \widetilde{v}(S, \varphi_{SS}),$$

so that $u^{\min}$ is a subsolution of (20). Similarly, $u^{\max}$ is a supersolution of that equation. It follows from Theorem 3.3 that under (A1)–(A3) the bounds $u^{\min} \leq u \leq u^{\max}$ hold for the viscosity solution u of (20), (13).

3.4. Relation to convex risk measures. The next proposition shows that solutions of the terminal-boundary value problem (20), (13) satisfy the axioms of a convex measure of risk.

PROPOSITION 3.5. *Suppose that assumptions* (A1)–(A3) *hold. Consider two continuous terminal value claims h^0 and h^1, and denote by u^0 and u^1 the corresponding (viscosity) solutions of the boundary value problem* (20), (13). *Then the following hold:*

1. *Monotonicity. If $h^0 \le h^1$ on $\partial^* Q$, then $u^0(t,S) \le u^1(t,S)$ for $(t,S) \in \overline{Q}$.*
2. *Convexity. For $\gamma \in [0,1]$ denote by u^γ the solution of* (20), (13) *with boundary value $h^\gamma = (1-\gamma)h^0 + \gamma h^1$. Then $u^\gamma \le (1-\gamma)u^0 + \gamma u^1$.*

Proof. Under (A1)–(A3) the functions u^γ, $\gamma \in [0,1]$, are uniquely defined and equal to the value function of the control problem (21), (22). Hence they can be written in the form

$$(26) \qquad u^\gamma(t,S) = \sup \left\{ E_{t,S} \left(\tilde{h}^\gamma(\tau, S_\tau) - \int_t^\tau \frac{1}{2} S_s^2 v^*(S_s, \lambda_s)\, ds \right) : \Lambda \in \mathcal{U}^{[\underline{v}, \overline{v}]} \right\}.$$

Monotonicity is obvious from this representation. Denote the expectation on the right-hand side of (26) by $J^\gamma(t,S,\Lambda)$. Then convexity follows from the following chain of inequalities:

$$(1-\gamma)u^0(t,S) + \gamma u^1(t,S) = (1-\gamma) \sup_{\Lambda \in \mathcal{U}^{[\underline{v}, \overline{v}]}} J^0(t,S,\Lambda) + \gamma \sup_{\tilde{\Lambda} \in \mathcal{U}^{[\underline{v}, \overline{v}]}} J^1(t,S,\tilde{\Lambda})$$

$$\ge \sup \left\{ (1-\gamma)J^0(t,S,\Lambda) + \gamma J^1(t,S,\Lambda) : \Lambda \in \mathcal{U}^{[\underline{v}, \overline{v}]} \right\}$$

$$= \sup \left\{ J^\gamma(t,S,\Lambda) : \Lambda \in \mathcal{U}^{[\underline{v}, \overline{v}]} \right\},$$

and the last expression is obviously equal to $u^\gamma(t,S)$. ☐

Fix $(t,S) \in Q$, put $\mathcal{H} = \{h : [\underline{S}, \overline{S}] \to \mathbb{R},\ h \text{ is continuous}\}$, and define a mapping $\varrho \colon \mathcal{H} \to \mathbb{R}$, $h \mapsto u^h(t,S)$, where u^h is the solution of (20) and (13). In view of our discussion in section 2, $\varrho(h)$ can be interpreted as the risk management cost of the position h. By Proposition 3.5 ϱ then satisfies the axioms of a convex measure of risk.[2] Interestingly, a corresponding dual representation in the sense of [11] can be read off directly from (26). For $\Lambda \in \mathcal{U}^{[\underline{v}, \overline{v}]}$ denote by Q^Λ the law of the process (21), and set $\mathcal{Q} := \{Q^\Lambda : \Lambda \in \mathcal{U}^{[\underline{v}, \overline{v}]}\}$. For $Q^\Lambda \in \mathcal{Q}$ define the penalty function

$$(27) \qquad\qquad \alpha(Q^\Lambda) = E_{t,S}^{Q^\Lambda} \left(\int_t^\tau \frac{1}{2} S_s^2 v^*(S_s, \lambda_s)\, ds \right).$$

Then ϱ can be written in the form

$$(28) \qquad\qquad \varrho_t(h) = \sup \left\{ E_{t,S}^{Q^\Lambda} \left(\tilde{h}(\tau, S_\tau) \right) - \alpha(Q^\Lambda) : Q^\Lambda \in \mathcal{Q} \right\}.$$

Note that for ϱ to be coherent in the sense of [1] the penalty function α—and hence the conjugate function v^*—needs to take its values in the set $\{0, \infty\}$. This holds true for the function v^{uv} in (16) from the uncertain volatility model. The situation is different for the illiquid market models and the associated functions v^{CJP} and v^{reac}: here the range of the conjugate function v^* does contain strictly positive but finite values, so that ϱ is convex but not coherent. This is natural from an economic point of view. In fact, a major motivation for considering convex but noncoherent risk measures is the observation that the axiom of positive homogeneity is hard to defend on illiquid markets.

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[2]Translation invariance is obvious as for $c \in \mathbb{R}$ the solution of (20) with boundary value $\tilde{h} + c$ is obviously equal to $u^h + c$.

4. Asymptotic properties. In this section we study the asymptotics of solutions of (20) as the market frictions modeled by the function $\widetilde{v}$ become "large." Fix some payoff $h\colon [\underline{S}, \overline{S}] \to \mathbb{R}$, and consider a sequence of modified Black–Scholes equations

$$(29) \qquad u_t^{(n)} + \frac{1}{2}S^2\widetilde{v}^{(n)}(S, u_{SS}^{(n)}) = 0 \ \ \text{with} \ u_{|\partial^*Q}^{(n)} = h\,.$$

We assume that $\widetilde{v}^{(n)}$ is of the form (19) (but with $\underline{v}$, $\overline{v}$, and v^* depending on n). Our key assumption is the following.

(A4) The sequence $\widetilde{v}^{(n)}$ is increasing and converges pointwise to

$$v^{\infty}(S, q) := \begin{cases} 0\,, & q \leq 0, S \in [\underline{S}, \overline{S}], \\ \infty\,, & q > 0, S \in [\underline{S}, \overline{S}]. \end{cases}$$

Assumption (A4) is satisfied if we consider the uncertain volatility model with widening volatility bounds $\underline{\sigma}_n \downarrow 0$ and $\overline{\sigma}_n \uparrow \infty$. In the illiquid market models, that is, for $v = v^{\text{reac}}$ or $v = v^{\text{CJP}}$, assumption (A4) holds if we consider sequences $\rho_n \uparrow \infty$ (increasing price impact of the large trader), $\underline{v}_n \downarrow 0$, and $\overline{v}_n \uparrow \infty$.

We denote by h^{conc} the concave envelope of the payoff h, that is, the smallest concave function greater than h on $[\underline{S}, \overline{S}]$. Formally,

$$h^{\text{conc}}(S) = \min\{f(S) \mid f \colon [\underline{S}, \overline{S}] \to \mathbb{R} \text{ concave and } f \geq h\}$$

$$(30) \qquad = \min\{c \in \mathbb{R} \colon \exists \alpha \in \mathbb{R} \text{ with } c + \alpha(\tilde{S} - S) \geq h(\tilde{S}) \text{ for all } \tilde{S} \in [\underline{S}, \overline{S}]\}\,;$$

the equivalence of both characterizations follows from a separation theorem for convex sets. Note that by (30), h^{conc} gives the minimal cost of a static (buy and hold) strategy that superreplicates the payoff h. Now we have the following theorem.

THEOREM 4.1. *Under* (A1)–(A4) *the sequence $u^{(n)}$ is increasing with* $\lim_{n\to\infty} u^{(n)} = h^{\text{conc}}$.

In economic terms the theorem states that for "large market frictions," such as a very strong price impact of the option hedger or very wide volatility bounds, the solution $u^{(n)}$ of the modified Black–Scholes equation (29)—which can be seen as *dynamic* hedge cost of the claim h—converges to the cost of the cheapest *static* replication strategy. A related result has been established by [9] in the context of superhedging in stochastic volatility models, and our proof uses arguments similar to theirs. A graphical illustration of Theorem 4.1 for the case of a call-spread is given in Figure 2.

Proof. Without loss of generality we may assume that $h \geq 0$ and hence also $u^{(n)} \geq 0$. The sequence $u^{(n)}$ is increasing as $\widetilde{v}^{(n)}$ is increasing; this follows from Theorem 3.3 (the comparison principle for solutions of the modified Black–Scholes equation) by an argument similar to that in Example 3.4. Moreover, the function $\kappa(t, S) := h^{\text{conc}}(S)$ is a supersolution of (29) for n fixed, as $\widetilde{v}^{(n)}(S, q) \leq 0$ for $q \leq 0$. Again by Theorem 3.3 we thus have $u^{(n)}(t, S) \leq h^{\text{conc}}(S)$ for all $t \in [0, T], S \in [\underline{S}, \overline{S}]$. Define

$$(31) \qquad u^{\infty}(t, S) := \lim_{n\to\infty} u^{(n)}(t, S) \quad \text{and} \quad \underline{u}^{\infty}(t, S) := \liminf_{\substack{n\to\infty, \\ (\tilde{t}, \tilde{S})\to(t, S)}} u^{(n)}(\tilde{t}, \tilde{S})\,.$$

We obviously have the inequalities $\underline{u}^{\infty} \leq u^{\infty} \leq h^{\text{conc}}$. In Lemmas 4.2 and 4.3 below we will show that $\underline{u}^{\infty}$ is concave in S for fixed t and nonincreasing in t. Moreover,

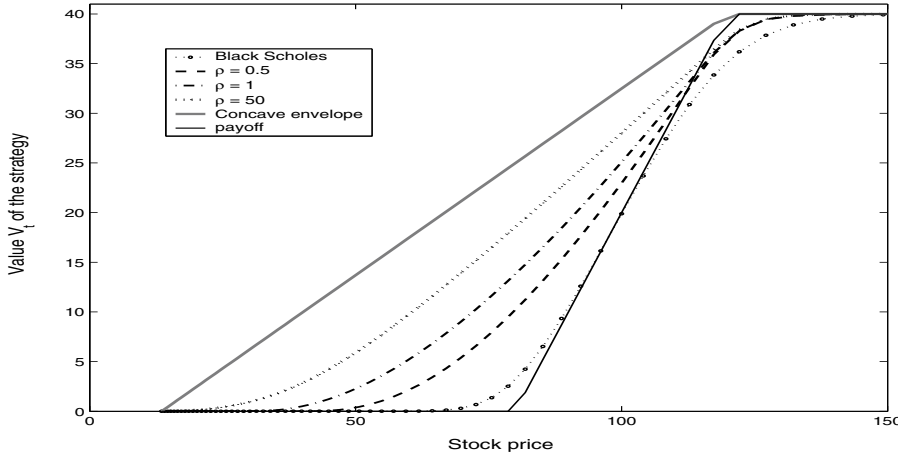$$\underline{u}^{\infty}(T, S) \geq \liminf_{(t, \tilde{S})\to(T, S)} u^{(1)}(t, S) = h(S)$$

FIG. 2. *Solutions of PDE* (29) *(for $v = v^{reac}$ and varying values of ρ) in comparison to the concave envelope and to the payoff function for the case of a European call-spread with strikes $K_1 = 80$ and $K_2 = 120$, i.e., $h(S) = [S - K_1]^+ - [s - K_2]^+$. The initial stock price is 100, and the time to maturity is six months.*

by the monotonicity of the sequence $u^{(n)}$. Hence we have $\underline{u}^\infty(T, S) \geq h^{\text{conc}}$. Combining the two inequalities gives $\underline{u}^\infty = u^\infty = h^{\text{conc}}$, and the theorem is proved. □

LEMMA 4.2. *The function $\underline{u}^\infty(t, \cdot)$ is concave in S for fixed t.*

Proof. In the first step of the proof we show that $\underline{u}^\infty$ is a supersolution of the equation $-u_{SS} = 0$. Fix n_0 and note that $u^{(m)}$ is a supersolution of (29) for every $m > n_0$. Theorem A.1 in the appendix (a stability result for viscosity solutions from [4]) shows that $\underline{u}^\infty$ is also a supersolution of (29) for every $n = n_0$ and, as n_0 was arbitrary, for every n. Assume now that $\underline{u}^\infty$ is not a supersolution of the equation $-u_{SS} = 0$. Then there exist a point $(t, s) \in [0, T) \times [\underline{S}, \overline{S}]$ and a test function $\varphi \in \mathcal{C}^{1,2}$ with $(\varphi_t, \varphi_S, \varphi_{SS}) \in \mathcal{P}^{2,-}(\underline{u}^\infty(t, S))$ (the subjet of $\underline{u}^\infty$ in the point (t, S)), so that $-\varphi_{SS}(t, S) < 0$. Using assumption (A4), we have for n sufficiently large

$$(32) \qquad \varphi_t(t, S) + \frac{1}{2} S^2 \widetilde{v}^{(n)}(S, \varphi_{SS}(t, S)) > 0,$$

contradicting the fact that $\underline{u}^\infty$ is a supersolution of (29) for every n.

Now we turn to the concavity of $\underline{u}^\infty$. By [9, Lemma 4.1], the function $\underline{u}^\infty(t, \cdot)$ is also a viscosity supersolution of $-u_{SS} = 0$ for fixed t. Now we fix t and a, b with $\underline{S} \leq a < b \leq \overline{S}$. Consider for $\delta > 0$ the boundary value problem

$$(33) \quad \delta u - u_{SS} = 0, \ S \in (a, b), \quad \text{with } u(a) = \underline{u}^\infty(t, a) \quad \text{and} \quad u(b) = \underline{u}^\infty(t, b).$$

Since $\underline{u}^\infty \geq 0$, it is a viscosity supersolution of the equation $u = 0$, and by the first step it is also a supersolution of (33) for every $\delta > 0$. Following [9], a subsolution of (33) is given by

$$H[\delta](S) = \frac{\underline{u}^\infty(t, a)[e^{\sqrt{\delta}(b-S)} - 1] + \underline{u}^\infty(t, b)[e^{\sqrt{\delta}(S-a)} - 1]}{e^{\sqrt{\delta}(a-b)} - 1}.$$

A general comparison theorem for viscosity solutions such as Theorem 3.3 in [8] provides the relation $\underline{u}^\infty(t, S) \geq H[\delta](S)$ for all $\delta > 0$. Setting $S = \lambda a + (1 - \lambda)b$ for

some $\lambda \in [0,1]$ and sending δ to zero, we obtain

$$\underline{u}^\infty(t, \lambda a + (1-\lambda)b) \geq \lambda \underline{u}^\infty(t,a) + (1-\lambda)\underline{u}^\infty(t,b),$$

as claimed. □

LEMMA 4.3. *The function $\underline{u}^\infty(\cdot, S)$ is decreasing in t.*

Proof. First, we show that the function $\underline{u}^\infty$ is a viscosity supersolution of the equation $-u_t = 0$. Assume to the contrary that there exists a point (t,S) and a test function $\varphi \in \mathcal{C}^{1,2}$ with $(\varphi_t, \varphi_S, \varphi_{SS}) \in \mathcal{P}^{2,-}(\underline{u}^\infty(t,S))$ and $\varphi_t(t,S) > 0$. We consider the expression

$$b := \varphi_t(t,S) + \frac{1}{2}S^2 \widetilde{v}^{(n)}\big(S, \varphi_{SS}(t,S)\big).$$

Since $\varphi_t(t,S) > 0$, using assumption (A4) we can choose n sufficiently large such that $b > 0$, which contradicts the fact that $\underline{u}^\infty(t,s)$ is a viscosity supersolution of (29). As before, [9, Lemma 4.1] shows that the function $\underline{u}^\infty(\cdot, S)$ is also a viscosity supersolution of $-u_t = 0$ for constant S. Now consider for arbitrary $0 \leq t_1 < t_2 \leq T$ and fixed S the terminal value problem $-u_t = 0$, $u(t_2) = \underline{u}^\infty(t_2, S)$. The constant function $c(t) = \underline{u}^\infty(t_2, S)$, $t \in [t_1, t_2]$, is a solution of the equation. Theorem 3.3 in [8] yields the inequality $c(t) \leq \underline{u}^\infty(t,S)$ and in particular $\underline{u}^\infty(t_2, S) = c(t_1) \leq \underline{u}^\infty(t_1, S)$. □

5. Application to models for illiquid markets. As mentioned before, for the functions v^{CJP} and v^{reac} introduced in (14) and (15) the equality $v(S,q) = \widetilde{v}(S,q)$ holds only if $|q|$ is not too large relative to the liquidity parameter ρ. Hence the results from the previous section—which were derived for the modified Black–Scholes equation governed by $\tilde{v}$—have to be applied with some care. In this section we study this issue in more detail.

5.1. Existence of classical solutions. We begin by discussing the relation between v and $\widetilde{v}$ for $v = v^{\text{CJP}}$ and $v = v^{\text{reac}}$. Since in both cases the mapping $q \mapsto v_q(S,q;\rho)$ is strictly increasing and continuous, we get from elementary calculus that for $\rho > 0$ fixed,

$$v(S,q) = v_q(S,q)q - v^*\big(S, v_q(S,q)\big).$$

It follows that the supremum in the dual representation (18) is attained at $\lambda = v_q(S,q)$, so that $v(S,q) = \widetilde{v}(S,q)$ for all q with $v_q(S,q) \in [\underline{v}, \overline{v}]$. On the other hand, for q_1 and q_2 with $v_q(S,q_1) < \underline{v}$, respectively, $v_q(S,q_2) > \overline{v}$, we have

$$\widetilde{v}(S,q_1) = q_1\underline{v} - v^*(S,\underline{v}) < v(S,q_1) \text{ and } \widetilde{v}(S,q_2) = q_2\overline{v} - v^*(S,\overline{v}) < v(S,q_2).$$

Note that this implies in particular that $\widetilde{v}_q(S,q) \in [\underline{v}, \overline{v}]$ for all $q \in \mathbb{R}$ and all S. A graphical illustration of the relation between v and $\widetilde{v}$ is given in Figure 1 in section 3.

Now for $v = v^{\text{CJP}}$ and $v = v^{\text{reac}}$ one has $v_q(S,q;\rho) \to \sigma^2$ locally uniformly as $\rho \to 0$. Hence $v(S,q;\rho) = \widetilde{v}(S,q;\rho)$ if ρ and $|q|$ are sufficiently small. In the next proposition we use this fact to establish the existence of classical solutions of the original nonlinear Black–Scholes equation for small ρ. Related results have been obtained previously by [12] and [20].

PROPOSITION 5.1. *Fix two constants $0 < \underline{v} \leq \sigma^2 \leq \overline{v} < \infty$. Suppose that the assumptions of Theorem 3.1 are in force and that v is as in (14) or (15). Then for all ρ sufficiently small the solution u of the modified terminal-boundary value problem (20), (13) solves also the original equation $u_t + \frac{1}{2}S^2 v(S, u_{SS}; \rho) = 0$.*

Proof. Fix some constant M_1. According to Theorem 3.1, for all $\rho \geq 0$ and all constants $\underline{\lambda}, \overline{\lambda}$ such that $\underline{v} \leq \underline{\lambda} \leq \overline{\lambda} \leq \overline{v}$ and $\sup\left\{v^*(S, \lambda; \rho) \colon S, \lambda \in [\underline{S}, \overline{S}] \times [\underline{\lambda}, \overline{\lambda}]\right\} \leq M_1$, there is a classical solution of the PDE

$$(34) \qquad u_t + \frac{1}{2}S^2 \sup\left\{\lambda u_{SS} - v^*(S, \lambda; \rho) \colon \lambda \in [\underline{\lambda}, \overline{\lambda}]\right\} = 0$$

satisfying the boundary condition (13). Moreover, there is some M_2 such that for all such solutions, $\|u_{SS}\|_\infty \leq M_2$. Suppose now that we can find for ρ sufficiently small two constants $\underline{\lambda}(\rho) \leq \overline{\lambda}(\rho) \in [\underline{v}, \overline{v}]$ such that the following two conditions hold:

(i) $v_q(S, q; \rho) \in [\underline{\lambda}(\rho), \overline{\lambda}(\rho)]$ for all $|q| \leq M_2$ and all $S \in [\underline{S}, \overline{S}]$;

(ii) $v^*(S, \lambda; \rho) \leq M_1$ for all $\lambda \in [\underline{\lambda}(\rho), \overline{\lambda}(\rho)]$ and all $S \in [\underline{S}, \overline{S}]$.

Then $v(S, q; \rho) = \sup\left\{\lambda q - v^*(S, \lambda; \rho) \colon \lambda \in [\underline{\lambda}(\rho), \overline{\lambda}(\rho)]\right\}$ for all $|q| \leq M_2, S \in [\underline{S}, \overline{S}]$, so that the solution of (34) solves also the original equation (12).

Condition (i) obviously holds for ρ sufficiently small if we choose

$$\underline{\lambda}(\rho) := \inf_{S \in [\underline{S}, \overline{S}]} v_q(S, -M_2; \rho) \quad \text{and} \quad \overline{\lambda}(\rho) := \sup_{S \in [\underline{S}, \overline{S}]} v_q(S, M_2; \rho).$$

In order to establish (ii) we show that for $\rho \to 0$, $v^*\big(S, v_q(S, q; \rho); \rho\big)$ converges to zero uniformly on $[\underline{S}, \overline{S}] \times [-M_2, M_2]$. For this, note that

$$(35) \qquad v^*\big(S, v_q(S, q; \rho); \rho\big) = q v_q(S, q; \rho) - v(S, q; \rho).$$

A Taylor approximation around $\rho = 0$ shows that the right-hand side of (35) is of the form

$$(36) \qquad q\left(v_q(S, q; 0) + \rho v_{q\rho}(S, q; 0)\right) - \left(v(S, q; 0) + \rho v_\rho(S, q; 0)\right) + o(\rho).$$

Without loss of generality we put $\sigma \equiv 1$. A direct computation shows that for v as in (14) and (15) one has

$$v(S, q; 0) = q; \ v_\rho(S, q; 0) = 2Sq^2; \ v_q(S, q; 0) = 1; \ v_{q\rho}(S, q; 0) = 4Sq.$$

Plugging this into (36) we get

$$v^*\big(S, v_q(S, q; \rho); \rho\big) = q(1 + 4Sq) - (q + 2Sq^2) + o(\rho) = \rho 2Sq^2 + o(\rho).$$

Hence $v^*(S, v_q(S, q; \rho); \rho) \to 0$ for $\rho \to 0$, and the proof of the proposition is complete. □

Implications for hedging. In section 2 we have derived the nonlinear Black–Scholes equation for the illiquid market models using informal hedging arguments. From these arguments we can conclude that under the assumptions of Proposition 5.1, a standard delta hedge with hedge ratio $\phi_t = u_S(t, S_t)$ is a perfect replication strategy for ρ sufficiently small, where "sufficiently small" means that

$$(37) \qquad \underline{v} \leq \inf_{(t,S) \in Q} v_q(S, u_{SS}(t, S); \rho) \quad \text{and} \quad \sup_{(t,S) \in Q} v_q(S, u_{SS}(t, S); \rho) \leq \overline{v}.$$

While ρ is typically small (recall that [5] obtained a value of the order of 10^{-4}), condition (37) is hard to verify a priori as it depends also on $\|u_{SS}\|_\infty$. This quantity depends on, in turn, among others, the size and the degree of nonlinearity of the payoff h to be hedged. Results from numerical experiments in [14] indicate, however, that for typical payoffs and parameter values, violations of (37) arise at most if the time to maturity is very short. Hence from a practical point of view it appears reasonable to use a standard delta hedging strategy with $\phi_t = u_S(t, S_t)$ for risk management purposes.

5.2. Superhedging cost in the CJP-model. In the CJP-model it is not a priori clear that a solution of the original CJP equation (3) gives the minimal hedge cost for a terminal value claim h. In fact, it has been shown in [6] that in the model (1) any payoff can be hedged approximately by using continuous trading strategies of finite variation and that the associated minimal hedge cost is the standard Black–Scholes price of the claim.[3] However, the class of continuous hedging strategies of finite variation is not useful for practical trading. Using a narrower—and in fact more reasonable—class of admissible trading strategies [7] showed that under growth conditions the minimal superhedging cost for a claim h is given by the unique continuous viscosity solution u^{CJP} of the boundary value problem

$$(38) \qquad u_t + \frac{1}{2}S^2\widehat{v^{\mathrm{CJP}}}(S, u_{SS}) = 0 \text{ for } (t, S) \in Q; \quad u = \tilde{h} \text{ on } \partial^*Q.$$

Here $\widehat{v^{\mathrm{CJP}}}$ is the so-called *parabolic envelope* of v^{CJP}, that is, the largest increasing minorant of the function $v^{\mathrm{CJP}}(S, \cdot)$. Since v^{CJP} is convex, $\widehat{v^{\mathrm{CJP}}}$ is given by

$$(39) \qquad \widehat{v^{\mathrm{CJP}}}(S, q) = \sup\left\{\lambda q - (v^{\mathrm{CJP}})^*(S, \lambda)\colon \lambda \in [0, \infty)\right\}.$$

Moreover, [7] establish a comparison principle for (38).

Remark 5.2. The results of [7] have been obtained for a stock price in $(0, \infty)$, but we are certain that the results carry over to the simpler case of a bounded domain. In fact, the comparison principle for (38) on a bounded domain can be verified directly using Theorem V.8.1 of [10].

Consider now a sequence $\lambda_n \to \infty$, denote by $\tilde{v}^{(n)} = \sup\{\lambda q - (v^{\mathrm{CJP}})^*(S, \lambda)\colon \lambda \in [0, \lambda_n]\}$ the modified function associated with v^{CJP} via (19), and let $u^{(n)}$ be the solution of the corresponding modified Black–Scholes equation with boundary condition (13). Then we have the following lemma.

LEMMA 5.3. *The sequence $u^{(n)}$ converges monotonically to u^{CJP}.*

Proof. Obviously, the sequence $\tilde{v}^{(n)}$ converges pointwise monotonically to $\widehat{v^{\mathrm{CJP}}}$. Moreover, Theorem 3.3 (the comparison principle for the modified Black–Scholes equation) implies that the sequence $u^{(n)}$ is increasing; the comparison principle for (38) implies that $u^{(n)} \leq u^{\mathrm{CJP}}$. Denote by u^∞ the pointwise limit of the sequence $u^{(n)}$. Define

$$\underline{u}^\infty(t, S) = \liminf_{\substack{n \to \infty, \\ (\tilde{t}, \tilde{S}) \to (t, S)}} u^{(n)}(\tilde{t}, \tilde{S}) \text{ and } \overline{u}^\infty(t, S) = \limsup_{\substack{n \to \infty, \\ (\tilde{t}, \tilde{S}) \to (t, S)}} u^{(n)}(\tilde{t}, \tilde{S}).$$

Theorem A.1 in the appendix shows that $\underline{u}^\infty$ is a supersolution and $\overline{u}^\infty$ is a subsolution of (38). Since comparison holds for (38) we get that $\overline{u}^\infty \leq \underline{u}^\infty$. On the other hand, the definition of $\underline{u}^\infty$ and $\overline{u}^\infty$ gives the obvious inequality $\underline{u}^\infty \leq u^\infty \leq \overline{u}^\infty$. Combining these inequalities, we obtain that $\underline{u}^\infty = u^\infty = \overline{u}^\infty$, so that u^∞ is the unique viscosity solution of (38) and thus equal to u^{CJP}. ☐

By combining Lemma 5.3 and Proposition 3.5 we immediately get the following corollary (modulo the caveat of Remark 5.2).

COROLLARY 5.4. *In the CJP-model the superhedging price u^{CJP} satisfies the axioms of a convex measure of risk on the set of all continuous terminal value claims.*

We conjecture that analogous results can be obtained for the reaction-function models of section 2.2. However, to date there is no formal characterization of the superhedging price in these models available in the literature.

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[3]Bank and Baum [3] establish a similar result that also covers the reaction-function models from section 2.2.

6. Pricing relative to a book of derivatives. In this section we discuss the pricing of individual contracts in a book of derivatives given that the modified nonlinear Black–Scholes equation (20) is used in order to determine the risk management cost of the entire book. Consider a market maker trading in N different option contracts with payoff $h^i(S_T)$, $1 \le i \le N$, and suppose that his overall liability at some given time point $t < T$ is given by $h(S_T) = \sum_{i=1}^{N} n_i h^i(S_T)$. We suppose that the market maker uses the modified nonlinear Black–Scholes equation

$$(40) \quad u_t + \sup\left\{\frac{1}{2}S^2 \lambda u_{SS} - \frac{1}{2}S^2 v^*(S, \lambda) \colon \lambda \in [\underline{v}, \overline{v}]\right\} = 0 \text{ in } Q, \quad u = \tilde{h} \text{ on } \partial^* Q,$$

to measure the risk management cost associated with the liability. For technical reasons we moreover assume that $\underline{v} > 0$ (uniform parabolicity) and that v^* is smooth on $[\underline{S}, \overline{S}] \times [\underline{v}, \overline{v}]$. We denote the unique (viscosity) solution of (40) by u^h and put, as before, $\varrho(h) = u^h(t, S_t)$. As explained in section 2, u^h can be viewed as the cost of running a dynamic hedging strategy for the position h in an illiquid market or in a market with uncertain volatility, perhaps augmented by some additional safety provision.

In this context it is a priori unclear how the market maker should determine quotes for the constituents h^i of the portfolio in a way that takes into account the contribution of each contract to the overall risk management cost $\varrho(h)$. This difficulty arises since (20) is nonlinear, and thus the risk management cost of the entire position is not equal to the sum of the risk management cost of the individual contracts. In formal terms we are looking for a rule that sets the quotes $\boldsymbol{\pi}(h) = (\pi(h^1; h), \dots, \pi(h^N; h))'$, at which the market maker agrees to trade small quantities of the individual contracts given his current liability h.

We propose two properties for the pricing rule of the market maker. First, we assume that his pricing rule is *linear given the overall position*; i.e., we postulate that the price of a portfolio $\sum_{i=1}^{N} \gamma_i h^i$ is given by $\sum_{i=1}^{N} \gamma_i \pi(h^i; h)$, at least for $|\boldsymbol{\gamma}|$ small. This is essentially a consistency requirement that serves to rule out static arbitrage opportunities for counterparties of the market maker such as violations of put-call parity. Second, since the market maker has typically no information regarding the type of the next order (buy or sell order), it seems reasonable that he attempts to set his quotes $\boldsymbol{\pi}(h)$ in such a way that he is *indifferent* with respect to arbitrary small changes in his position. In order to formalize this idea we introduce the function

$$r(\cdot; h) \colon \mathbb{R}^N \to \mathbb{R}, \qquad \boldsymbol{\gamma} \mapsto \varrho\left(h + \sum_{i=1}^{N} \gamma_i h^i\right).$$

Now for given quotes $\boldsymbol{\pi}(h)$, selling the portfolio $\sum_{i=1}^{N} \gamma_i h^i$ leads to the additional income $\sum_{i=1}^{N} \gamma_i \pi(h^i; h)$, whereas the risk management cost changes from $\varrho(h)$ to $\varrho(h + \sum_{i=1}^{N} \gamma_i h^i)$. Hence the overall profit and loss (P&L) change of the deal is given by

$$\boldsymbol{\pi}(h)'\boldsymbol{\gamma} - \big(r(\boldsymbol{\gamma}; h) - r(0; h)\big).$$

Indifference with respect to small changes in the market maker's position thus suggests choosing $\boldsymbol{\pi}(h)$ as gradient of $r(\cdot; h)$ at $\boldsymbol{\gamma} = 0$. Unfortunately, $r(\cdot; h)$ is in general not differentiable (a counterexample is provided below). However, the convexity of ϱ established in Proposition 3.5 implies that the function $r(\cdot; h)$ is convex, so that its

subdifferential is nonempty. A feasible choice for the quote vector reflecting indifference as far as possible is therefore to take $\boldsymbol{\pi}(h)$ as a *subgradient* (an element of the subdifferential) of $r(\cdot; h)$ at $\boldsymbol{\gamma} = 0$.

The next lemma is the first step in computing a quote vector $\boldsymbol{\pi}(h)$. Related arguments are used in [18] to derive capital allocation principles in risk management.

LEMMA 6.1. *Consider a pair of processes* Λ^*, S^* *with* $\Lambda^* \in \mathcal{U}^{[\underline{v},\overline{v}]}$ *so that* S^* *has dynamics* $dS_t^* = \sqrt{\lambda_t^*} S_t^* dW_t$ *for some Brownian motion* W. *Suppose that the law* Q^* *of* S^* *solves the optimization problem* (28). *Then a subgradient of* $r(\cdot; h)$ *at* $\boldsymbol{\gamma} = 0$ *is given by* $\pi(h^i; h) = E_{t,S_t}^{Q^*}\big(\tilde{h}^i(\tau, S_\tau)\big)$, $1 \le i \le N$.

Proof. According to the dual representation (28) of the risk management cost ϱ we get for any $\boldsymbol{\gamma} \in \mathbb{R}^N$ that

$$r(\boldsymbol{\gamma}; h) \ge E_{t,S_t}^{Q^*}\left(\tilde{h}(\tau, S_\tau) + \sum_{i=1}^N \gamma_i \tilde{h}^i(\tau, S_\tau)\right) - \alpha(Q^*) = r(0; h) + \sum_{i=1}^N \gamma_i E_{t,S_t}^{Q^*}\big(\tilde{h}^i(\tau, S_\tau)\big),$$

where the last equality follows from the optimality of Q^*. Hence

$$\lim_{\alpha \to 0^+} \frac{1}{\alpha}\big(r(\alpha\boldsymbol{\gamma}; h) - r(0; h)\big) \ge \sum_{i=1}^N \gamma_i E_{t,S_t}^{Q^*}\big(\tilde{h}^i(\tau, S_\tau)\big) = \boldsymbol{\pi}(h)'\boldsymbol{\gamma},$$

which establishes the claim. $\square$

In the following theorem we use verification results in order to describe Λ^* and Q^*.

THEOREM 6.2. *Suppose that the modified Black–Scholes equation admits a solution* $u^h \in \mathcal{C}^{1,2}(Q) \cap \mathcal{C}(\overline{Q})$, *that* $\underline{v} > 0$, *and that* v^* *is smooth. Then the following hold:*

(i) *Consider a measurable function* $\alpha\colon Q \to [0,1]$ *and define the function*

$$\lambda^\alpha(t, S) = \alpha(t, S)\widetilde{v}_q^-\big(S, u_{SS}^h(t, S)\big) + (1 - \alpha(t, S))\widetilde{v}_q^+\big(S, u_{SS}^h(t, S)\big), \quad (t, S) \in Q.$$

Then there exists a weak solution $S^{*,\alpha}$ *of the stochastic differential equation* $dS_t = (\lambda^\alpha(t, S_t))^{\frac{1}{2}} S_t dW_t$, *and the law* $Q^{*,\alpha}$ *of* $S^{*,\alpha}$ *solves the optimization problem* (28), *so that the choice* $\pi(h^i; h) = E_{t,S_t}^{Q^{*,\alpha}}\big(\tilde{h}^i(\tau, S_\tau)\big)$ *defines a subgradient of* $r(\cdot; h)$ *at* $\boldsymbol{\gamma} = 0$.

(ii) *Suppose in addition that the function* $\widetilde{v}_q(S, q)$ *is locally Lipschitz in* $[\underline{S}, \overline{S}] \times \mathbb{R}$ *and that* u_{SS}^h *is locally Hölder-continuous in* Q. *Then there exists a solution of the linear boundary value problem*

$$(41) \quad u_t^i(t, S) + \frac{1}{2} S^2 \widetilde{v}_q\big(S, u_{SS}^h(t, S)\big) u_{SS}^i(t, S) = 0 \text{ in } Q, \quad u^i = \tilde{h}^i \text{ on } \partial^* Q,$$

and we have $\pi(h^i; h) = u^i(t, S_t)$.

Proof. By standard results on conjugate functions we obtain that

$$\lambda^\alpha(t, S) \in \operatorname{argmin}\left\{\frac{1}{2}\lambda S^2 u_{SS}^h(t, S) - \frac{1}{2} S^2 v^*(S, \lambda)\colon \lambda \in [\underline{v}, \overline{v}]\right\}.$$

In view of uniform parabolicity the existence of a weak solution $S^{*,\alpha}$ of the state equation and the optimality of the associated law then follows from Theorem IV.4.4 of [10] and the discussion preceding it. For (ii) note that under the regularity assumptions on u_{SS}^h and $\widetilde{v}_q$ the function $(t, S) \mapsto S^2 \widetilde{v}_q\big(S, u_{SS}^h(t, S)\big)$ is locally Hölder-continuous

on Q. The existence of a solution u^i to (41) then follows from Corollary 2 on page 71 of [16]. Moreover, since v_q is assumed to be continuous, $\lambda(t, S) = \widetilde{v}_q\big(S, u^h_{SS}(t, S)\big)$, independent of α. The relation

$$u^i(t, S) = E^{Q^*}_{t, S_t}\big(\tilde{h}^i(\tau, S_\tau)\big) = \pi(h^i; h)$$

is finally an immediate consequence of the Feynman–Kac formula.   □

Comments. 1. The solution u^i of (41) (and hence the quote $\pi(h^i; h)$) can be viewed as the price of the option h^i in a standard one-dimensional diffusion model with price-dependent volatility

$$\sigma^h(t, S) = \big(\widetilde{v}_q(S, u^h_{SS}(t, S))\big)^{\frac{1}{2}}.$$

Note that σ^h depends on the overall liability h of the market maker via u^h (except in the special case of the classical Black–Scholes model, where $v = \sigma^2 q$ so that $\widetilde{v}_q = \sigma^2$). It follows that the pricing principle $\boldsymbol{\pi}(h)$ tends to increase the price of options with convexity properties similar to the overall position h and to decrease the price of contracts with opposite convexity properties. Suppose for concreteness that $h(\cdot)$ is strictly convex and that $v = v^{\mathrm{CJP}}(S, q; \rho, \sigma)$ for $\rho > 0$. Then $u^h_{SS}(t, S) \geq 0$, and hence for $\overline{v}$ sufficiently large,

$$\widetilde{v^{\mathrm{CJP}}}_q(S, u^h_{SS}(t, S); \rho, \sigma) = \min\{\overline{v}, \sigma^2(1 + 2\rho S u^h_{SS}(t, S))\} > \sigma^2.$$

If h^i is convex (concave), a comparison argument gives that $\pi(h^i; h)$ is bigger (smaller) than the Black–Scholes price of h^i in a reference model with constant volatility σ.

2. If the liability h of the market maker is smooth (at least $\mathcal{C}^3$), the existence of a solution $u^h \in \mathcal{C}^{1,2}(Q) \cap \mathcal{C}(\overline{Q})$ with locally Hölder-continuous second derivative u^h_{SS} follows from Theorem 6.4.2 of [19]. A pragmatic way of using Theorem 6.2 is therefore to approximate h by a smooth function g and to use (41) with "squared volatility" $\widetilde{v}_q\big(S, u^g_{SS}(t, S)\big)$. A formal justification of this procedure is left for further research.

3. For a simple example where $r(\cdot; h)$ is not differentiable take $v = v^{\mathrm{uv}}$ with $\underline{\sigma} < \overline{\sigma}$ and assume that h is linear in S. Then $u^h_{SS} \equiv 0$ and $\lambda^\alpha(t, S) = (\alpha\underline{\sigma}^2 + (1 - \alpha)\overline{\sigma}^2)$ so that $Q^{*, \alpha}$ is the law of a geometric Brownian motion with volatility $(\alpha\underline{\sigma} + (1 - \alpha)\overline{\sigma})$. Suppose, moreover, that h^1 is a standard call option. Then we have for constants $0 \leq \alpha < \beta \leq 1$ that

$$E^{Q^{*,\alpha}}_{t, S_t}\big(\tilde{h}^i(\tau, S_\tau)\big) < E^{Q^{*,\beta}}_{t, S_t}\big(\tilde{h}^i(\tau, S_\tau)\big),$$

so that the subdifferential of $r(\cdot; h)$ at $\boldsymbol{\gamma} = 0$ contains more than one element.

7. Conclusion. In this paper we have studied properties of solutions to typical nonlinear Black–Scholes equations arising in derivative asset analysis in illiquid markets or in markets with uncertain volatility. Using duality results for conjugate functions it was observed that after a minor modification the equations can be viewed as a dynamic programming equation of an associated stochastic control problem. Existence and comparison results for this equation were established. Moreover, it was shown that the risk management cost modeled by these equations satisfies the axioms of a convex measure of risk, and a dual representation of this risk measure was given. We showed that for large market frictions the solution of typical nonlinear Black–Scholes equations converges to the concave envelope of the payoff. Finally, we

explained how the control problem associated with the nonlinear Black–Scholes equations can be used to determine prices for individual contracts in a book of derivatives in a consistent way.

Appendix. Stability of viscosity solutions. For the convenience of the reader we quote a stability result for viscosity solutions [4, Theorem 4.1], which is used at a number of points in the paper. Given a sequence of functions $u^{(n)} \colon \mathbb{R}^N \to \mathbb{R}$ define

$$\limsup{}^* u^{(n)}(x) = \limsup_{y \to x, n \to \infty} u^{(n)}(y) \quad \text{and} \quad \liminf{}_* u^{(n)}(x) = \liminf_{y \to x, n \to \infty} u^{(n)}(y).$$

THEOREM A.1. *Let* $F^n \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$ *be locally uniformly bounded and proper* ($\mathcal{S}(N)$ *denotes the set of all symmetric* $N \times N$ *matrices). Consider the equation*

(42) $$F^n(x, u, D_x u, D_x^2 u) = 0 \quad on \ \overline{\Omega} \subseteq \mathbb{R}^N.$$

Define $\underline{F} = \liminf_* F^n$ *and* $\overline{F} = \limsup^* F^n$. *Let* $u^{(n)}$ *be a sequence of locally bounded functions on* $\overline{\Omega}$. *Suppose that* $u^{(n)}$ *is a subsolution of* (42) *for every* n. *Then* $\overline{u} = \limsup^* u^{(n)}$ *is a subsolution of* $\underline{F}(x, u, D_x u, D_x^2 u) = 0$ *on* $\overline{\Omega}$. *Similarly, if* $u^{(n)}$ *is a supersolution of* (42) *for every* n, *then* $\underline{u} = \liminf^* u^{(n)}$ *is a supersolution of* $\overline{F}(x, u, D_x u, D_x^2 u) = 0$ *on* $\overline{\Omega}$.

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