VALUE ADJUSTMENTS AND DYNAMIC HEDGING OF REINSURANCE COUNTERPARTY RISK*

CLAUDIA CECI †, KATIA COLANERI‡, RÜDIGER FREY§, AND VERENA KÖCK¶

Abstract. Reinsurance counterparty credit risk (RCCR) is the risk of a loss arising from the fact that a reinsurance company is unable to fulfill her contractual obligations towards the ceding insurer. RCCR is an important risk category for insurance companies which, so far, has been addressed mostly via qualitative approaches. In this paper we therefore study value adjustments and dynamic hedging for RCCR. We propose a novel model that accounts for contagion effects between the default of the reinsurer and the price of the reinsurance contract. We characterize the value adjustment in a reinsurance contract via a partial integro-differential equation (PIDE) and derive the hedging strategies using a quadratic method. The paper closes with a simulation study which shows that dynamic hedging strategies have the potential to significantly reduce RCCR.

Key words. Reinsurance, Counterparty Risk, Credit Value Adjustment, Quadratic Hedging

AMS subject classifications. 91G40, 60J75, 60J60.

1. Introduction.

General insurers frequently cede parts of their insurance risk to reinsurance companies in order to protect themselves from intolerably large losses in their insurance portfolio. This gives rise to a new type of risk, so-called *reinsurance counterparty credit risk* or RCCR. This is the risk of a loss for the ceding company caused by the fact that the reinsurer fails to honor her obligations from a reinsurance contract, for instance because the reinsurer defaults prior to maturity of the contract. Given the increased visibility of default risk in the reinsurance industry in the aftermath of the financial crisis, RCCR has become a highly relevant risk category, mainly because reinsurance recoveries represent large assets on insurance companies balance sheets. Its importance is also underlined in Solvency II regulatory directives. Nonetheless, the techniques for managing RCCR used in practice are mostly of a qualitative nature. Typically, ceding companies have minimum requirements on the credit quality of approved reinsurance companies, they set limits for the exposure to individual counterparties, and they sometimes require reinsurers to post some collateral; see for instance [6]. The existing quantitative approaches for the management of RCCR are based on simple one-period models. This is in stark contrast to the banking world where sophisticated stochastic models are used in counterparty risk management to determine value adjustments for derivative transactions (so-called XVAs) and to find dynamic hedging and collateralization strategies, see for instance [23] or [9] for an overview.

In this paper we explore the potential of dynamic risk management techniques for reinsurance counterparty risk. Our objective is twofold: we discuss the computation of value adjustments to account for reinsurance default when pricing a contract, and we analyse dynamic hedging strategies in view of reducing the risk exposure. In fact,

*Submitted to the editors DATE.

†Department of Economics, University "G. D'Annunzio" of Chieti-Pescara, Viale Pindaro, 42, 65127 Pescara, Italy. (c.ceci@unich.it)

‡Department of Economics and Finance, University of Rome Tor Vergata, Via Columbia, 2, 00133 Rome, Italy (katia.colaneri@uniroma2.it).

§Institute for Statistics and Mathematics, WU-University of Economics and Business, Welthandelsplatz 1, 1020, Vienna, Austria (rfrey@wu.ac.at).

¶Institute for Statistics and Mathematics, Vienna University of Economics and Business, Welthandelsplatz, 1, 1020 Vienna, Austria (verena.koeck@wu.ac.at).

counterparty risk towards a major reinsurance company is a low-frequency, high-severity event so that bearing this risk is not attractive for the ceding company.

We consider a setting that is tailored to the analysis of RCCR. We model the aggregate claim amount process L underlying the reinsurance contract under consideration by a doubly stochastic compound Poisson process. To capture the effect that "reinsurance companies are most likely to default in times of market stress, that is exactly when cedants are most reliant upon their reinsurance covers" (see [19]), we introduce several sources of dependence between the aggregate claim amount L and the default process H^R of the reinsurance company. There is positive correlation between the claim arrival intensity λ^L and the default intensity λ^R of the reinsurer; moreover, λ^L exhibits a contagious jump at the default time τ_R of the reinsurer. In line with the concept of market consistent valuation we define the credit value adjustment (CVA) for a reinsurance contract as the expected discounted value of the replacement cost for the contract incurred by the insurer at the default time τ_R. Using mathematical results from the companion paper [14], we characterize the CVA as classical solution of a partial integro-differential equation. Next we address the hedging of RCCR by dynamic trading in a credit default swap (CDS) on the reinsurance company. Here we resort to a quadratic hedging approach (see [28]), since perfect replication is not possible. To determine the hedging strategy we make use of an orthogonal decomposition of the CVA into a hedgeable and a non-hedgeable part, based on the Galtchouk-Kunita-Watanabe decomposition of the associated discounted gains process. The paper closes with a simulation study. We analyse the impact of model parameters on the size of the CVA and we compare the performance of various hedging strategies. Our numerical experiments show that dynamic CDS hedging strategies significantly reduce reinsurance counterparty risk, both compared to a static hedging strategy (a strategy where the CDS position is not adjusted) and to the case where the insurance company does not hedge at all. More generally, the results suggest that dynamic risk-mitigation techniques can be very useful tools in the management of reinsurance counterparty risk.

We continue with a discussion of the existing literature. The quantitative literature on RCCR is relatively scarce. Interesting contributions from practitioners include [29] or [19] who propose a static model to assess the distribution of the RCCR loss, which can be used for reserving and economic capital purposes. They employ corporate bonds and CDSs to estimate reinsurance default rates and model correlation between defaults by reinsurers' asset return correlations. Another example is offered by [24] who study the problem of optimising the weight of different reinsurance companies in a given reinsurance program in order to minimize the expected loss due to RCCR. Also the solvency capital requirement for RCCR under the Solvency II standard formula is computed from a simple one-period credit risk model, see for instance [13]. On the academic side, [2] and [10] study how the possibility of a default of the reinsurer affects the form of optimal reinsurance contracts. An excellent overview of counterparty risk management in banking is given in [23] or [9]. Other recent contributions are, for instance [15, 16, 5]. Quadratic hedging criteria such as mean variance hedging and risk minimization have been applied in the insurance framework mainly for hedging life insurance contracts (e.g. unit linked contracts). Some recent references are, for instance [26, 17, 30, 11, 3, 12]

The rest of the paper is organized as follows. In Section 2 we introduce and develop the modelling framework and discuss the different forms of interaction between the insurance and the reinsurance companies that are captured by our setting. A rigorous construction of the model dynamics is provided in Section 2.3. In Section 3 we discuss the price of the reinsurance contract and the value adjustment to account for the reinsurer default. The hedging problem is studied in Section 4, and Section 5 contains the results from the numerical analysis. Some longer computations are relegated to the Appendix.

2. The Model.

2.1. The Setup. We work on a measurable space $(\Omega, \mathcal{G})$ with a complete and right continuous filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$. We assume that on this space there are two equivalent probability measures: the physical measure $\mathbf{P}$ and a risk neutral measure $\mathbf{Q}$ which is used for the valuation of financial and actuarial contracts. Using a risk-neutral measure for pricing purposes is in line with the principle of *market consistency valuation*, which is frequently used in the insurance framework and which represents one of the core elements of the Solvency II regulatory regime.

We consider a setup with two companies: an insurance company, labelled I, and a reinsurer R, who enter into a reinsurance contract with a given maturity T (typically one year). To model the losses in the insurance portfolio underlying this contract we consider a sequence $\{T_n\}_{n\in\mathbb{N}}$ of claim arrival times and a sequence $\{Z_n\}_{n\in\mathbb{N}}$ of claim sizes. Precisely, the T_n are $\mathbb{G}$-stopping times such that $T_n < T_{n+1}$ a.s. and Z_n are a.s. strictly positive $\mathcal{G}_{T_n}$-measurable random variables. We define the counting process $N = (N_t)_{t\geq 0}$ by $N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq t\}}$, for every $t \geq 0$. Then the process $L = (L_t)_{t\geq 0}$ given by

$$L_t = \sum_{n=1}^{N_t} Z_n\,, \quad t \geq 0,$$

describes the aggregate claim amount underlying the reinsurance contract. It will be convenient to work with the integer-valued random measure m^L on $\mathbb{R}^+ \times \mathbb{R}^+$ associated with the marked point process L, that is

$$m^L(\mathrm{d}t, \mathrm{d}z) = \sum_{n\geq 1} \delta_{\{T_n, Z_n\}}(\mathrm{d}t, \mathrm{d}z)\mathbf{1}_{\{T_n<\infty\}},$$

where $\delta_{\{t,z\}}$ is the Dirac measure at point $(t, z) \in \mathbb{R}^+ \times \mathbb{R}^+$. This allows for the following equivalent expression of L

$$L_t = \int_0^t \int_{\mathbb{R}^+} z\; m^L(\mathrm{d}s, \mathrm{d}z), \quad t \geq 0.$$

In our setting the reinsurance company may default and we denote by τ_R the $\mathbb{G}$-stopping time representing its default time; the default indicator process $H^R = (H^R_t)_{t\geq 0}$ is given by

$$H^R_t = \mathbf{1}_{\{\tau_R \leq t\}}, \quad t \geq 0.$$

If $\tau_R \leq T$, the reinsurer will not be able to fulfill his obligations which creates reinsurance counterparty credit risk (RCCR).

Next we specify the model for the loss process L and the default indicator H^R. In our analysis we are mostly concerned with valuation issues so we work under the risk-neutral measure $\mathbf{Q}$; to simplify the exposition we therefore introduce directly the $\mathbf{Q}$ dynamics of L and H^R. Model calibration and the relation between the measures $\mathbf{P}$ and $\mathbf{Q}$ are discussed in more detail in Section 2.2. We assume that the point process N modeling the claim arrivals has the $(\mathbb{G}, \mathbf{Q})$-intensity λ^L for a nonnegative $\mathbb{G}$-adapted cádlág process $\lambda^L = (\lambda^L_t)_{t\geq 0}$ (called in the sequel *loss intensity*), that is $(N_t - \int_0^t \lambda^L_{s-}\mathrm{d}s)_{t\geq 0}$ is a $(\mathbb{G}, \mathbf{Q})$-martingale. We assume that claim sizes are independent random variables with identical distribution $\nu(\mathrm{d}z)$, and also independent of N. Therefore the $(\mathbb{G}, \mathbf{Q})$-predictable compensator of the measure $m^L(\mathrm{d}t, \mathrm{d}z)$ is given by $\lambda^L_{t-}\nu(\mathrm{d}z)\mathrm{d}t$ [1]. We assume that the default indicator process H^R admits a stochastic intensity $\lambda^R = (\lambda^R_t)_{t\geq 0}$ (in the sequel called the *default intensity* of R), which is a nonnegative $\mathbb{G}$-adapted cádlág process such that the process

$$M^R_t := H^R_t - \int_0^{t\wedge\tau_R} \lambda^R_{s-}\mathrm{d}s, \quad t \geq 0, \tag{1}$$

is a $(\mathbb{G}, \mathbf{Q})$-martingale. Finally we describe the dynamics of the default and the claim arrival intensity. We assume that there is a standard two-dimensional $(\mathbb{G}, \mathbf{Q})$-Brownian motion $W = (W^1_t, W^2_t)_{t\geq 0}$ and that the processes λ^L and λ^R are of the form $\lambda^L_t = \lambda^L(X_t)$, $\lambda^R_t = \lambda^R(Y_t)$, $t \geq 0$, where $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are intensity-factor processes that satisfy the following system of SDEs

$$\mathrm{d}X_t = \gamma^X(X_{t-})\,\mathrm{d}H^R_t + b^X(X_t)\mathrm{d}t + \sigma^X(X_t)\mathrm{d}W^1_t, \quad X_0 = x_0 \in \mathbb{R},$$
$$\mathrm{d}Y_t = b^Y(Y_t)\mathrm{d}t + \sigma^Y(Y_t)(\rho\mathrm{d}W^1_t + \sqrt{1-\rho^2}\mathrm{d}W^2_t), \quad Y_0 = y_0 \in \mathbb{R},$$

for some $\rho \in [0, 1]$ and measurable functions $b^X, b^Y : \mathbb{R} \to \mathbb{R}$, $\sigma^X, \sigma^Y : \mathbb{R} \to \mathbb{R}^+$. We assume that the functions $\lambda^L : \mathbb{R} \to \mathbb{R}^+$ and $\lambda^R : \mathbb{R} \to \mathbb{R}^+$ and $\gamma^X : \mathbb{R} \to \mathbb{R}^+$ are continuous and increasing. A detailed construction of this model is given in Section 2.3. Modelling λ^L and λ^R as functions of the intensity factors X and Y is mathematically convenient, as it facilitates the application of mathematical results from the companion paper [14].

We assume that the indemnity payment of the reinsurance contract is of the form $\phi(L_T)$ for some bounded, increasing and Lipschitz continuous function ϕ. This covers typical forms of reinsurance (see, e.g. [1]). For examples, for a *stop loss reinsurance* contract with priority or lower attachment point $\underline{K}$ and upper limit $\overline{K}$ one takes $\phi(l) = \min\left\{\overline{K}, [l - \underline{K}]^+\right\}$ (with $[x]^+ = \max\{x, 0\}$). Another example is offered by the *excess-of-loss* (XL) contract with retention level M and upper limit $\overline{K}$. The payoff of this contract is given by

$$\min\Big\{\overline{K}, \sum_{n=1}^{N_T}[Z_i - M]^+\Big\}.$$

This can be written in the form $\phi(L^{\mathrm{XL}}_T)$ if we set $L^{\mathrm{XL}}_t = \sum_{\{T_n\leq t, Z_n > M\}}[Z_n - M]$ and $\phi(l) = \min\{\overline{K}, l\}$.

We denote by $r \geq 0$ the risk-free interest rate which is taken constant for simplicity. In line with market consistent valuation we define the *market value* of the reinsurance contract by

$$V^\phi_t := \mathbb{E}^{\mathbf{Q}}\left[e^{-r(T-t)}\phi(L_T)|\mathcal{G}_t\right], \quad 0 \leq t \leq T\,. \tag{2}$$

[1] By definition of $(\mathbb{G}, \mathbf{Q})$-predictable compensator, for every nonnegative, $\mathbb{G}$-predictable random function $(\Gamma(t, z))_{t\geq 0}$ with $\mathbb{E}^{\mathbf{Q}}\left[\int_0^t\int_{\mathbb{R}^+}|\Gamma(s, z)|\lambda^L_{s-}\nu(\mathrm{d}z)\mathrm{d}s\right] < \infty$, for every $t \geq 0$, the process

$$\int_0^t\int_{\mathbb{R}^+}\Gamma(s, z)(m^L(ds, dz) - \lambda^L_{s-}\nu(\mathrm{d}z)\mathrm{d}s), \quad t \geq 0,$$

is a $(\mathbb{G}, \mathbf{Q})$-martingale.

The quantity V_t^ϕ is the theoretical value of the reinsurance contract at time t. Due to the fact that the reinsurer R may default, the transaction price (the price at which I and R are actually entering into the contract) needs to be adjusted. This is done via the credit value adjustment introduced in Section 3.

Our setting accounts for various forms of dependence between the aggregate claim amount L and the default time τ_R. First, there is correlation between Brownian motions driving the intensities λ^L and λ^R, modelled by the parameter ρ. In practice one would take $\rho > 0$, so that in a scenario where the insurance company experiences many losses (high claim arrival intensity λ^L), the economic outlook for the reinsurance company gets less favourable (high default intensity λ^R). This models the observation that "often there are strong correlations between reinsurance default and the loss experience of the ceded portfolio" (see [19]). Second, there is *pricing contagion*: For $\gamma^X > 0$, the risk-neutral claim arrival intensity λ^L jumps upward at τ_R which translates into an upward jump of the market value V_t^ϕ of the reinsurance contract at $t = \tau_R$. Pricing contagion reflects the fact that the default of R reduces the supply for reinsurance (as R leaves the market), so that the insurer has to pay a higher price to renew his reinsurance cover. Note that each of these two forms of dependence between L and τ_R imply that

$$\mathbb{E}^{\mathbf{Q}}\left[V_t^\phi | \tau_R = t\right] > \mathbb{E}^{\mathbf{Q}}\left[V_t^\phi\right]. \tag{3}$$

Following the financial literature on counterparty risk we refer to this inequality as *wrong-way risk*.

We now introduce a set of assumptions that give sufficient conditions for existence and uniqueness for the solutions of certain partial integro-differential equations that arise in the computation of the value adjustment and of the hedging strategy. Define the instantaneous covariance matrix of (X, Y) as

$$\Sigma(x,y) := \begin{pmatrix} (\sigma^X(x))^2 & \rho\sigma^X(x)\sigma^Y(y) \\ \rho\sigma^X(x)\sigma^Y(y) & (\sigma^Y(y))^2 \end{pmatrix} \quad \text{for every} \quad (x,y) \in \mathbb{R}^2.$$

ASSUMPTION 2.1. *(A1) The functions b^X, b^Y, σ^X and σ^Y are Lipschitz;*

(A2) There exists $\beta > 0$ such that for every $w \in \mathbb{R}^2$ we have

$$w^\top \Sigma(x,y) w \geq \beta \|w\|^2.$$

(A3) The functions λ^L, λ^R are Lipschitz continuous and bounded.

(A4) The claim-size distribution ν has finite second moment.

2.2. Calibration. We now sketch an approach for the calibration of our model. This should also help to clarify the role played by the valuation measure $\mathbf{Q}$ as opposed to the historical measure $\mathbf{P}$. We begin with the calibration of the risk-neutral default intensity λ^R. In practice one would calibrate a model for λ^R to CDS spreads of R observed on the market, see e.g. [8, Chapter 22] for a detailed discussion and numerical examples. Here one is dealing only with market quantities, so that it is sufficient to consider the $\mathbf{Q}$-default intensity of R.

Next we describe a method for calibrating the $\mathbf{Q}$-characteristics of the loss process and we explain how to relate market consistent valuation of the reinsurance contract to more standard actuarial valuation approaches. At this point we are dealing with risks that are largely non-traded so that the $\mathbf{P}$-dynamics of the loss process are relevant as well. We proceed in three steps. In step one techniques from actuarial statistics (for instance [1, Chapter 4 and 5]) are used to estimate a doubly stochastic compound Poisson process model for L with claim size distribution $\nu^{\mathbf{P}}$ and loss intensity $\lambda^{L,\mathbf{P}}$ using historical loss data. In the estimation we propose to work with a model of the form $\lambda_t^{L,\mathbf{P}} = \lambda^{L,\mathbf{P}}(\widetilde{X}_t)$ for the $\mathbf{P}$-loss intensity and we assume that $\widetilde{X}$ follows a diffusion,

$$\mathrm{d}\widetilde{X}_t = b^X(\widetilde{X}_t)\mathrm{d}t + \sigma^X(\widetilde{X}_t)\mathrm{d}W_t^1\,, \quad \widetilde{X_0} = x_0\,,$$

for a $\mathbf{P}$-Brownian motion W^1. Notice that for the estimation we refer to a process $\widetilde{X}$ (see Section 2.3 below) which represents a contagion-free version of the intensity-factor X (i.e. the processes X and $\widetilde{X}$ share the same dynamics up to time τ_R but $\widetilde{X}$ does not jump at τ_R). This reflects the fact that the default of R has no impact on the $\mathbf{P}$-loss intensity as $\lambda^{L,\mathbf{P}}$ models the arrival intensity of claim events in the real world such as storms or flooding. The reinsurance contract is then valued via an actuarial premium principle, see e.g. [1, Chapter 7], leading to the counterparty-risk free *actuarial value* of the contract. For instance, one could use the expected value principle with safety-loading parameter $\alpha > 0$, which gives an actuarial value equal to $(1+\alpha)\mathbb{E}^{\mathbf{P}}[e^{-rT}\phi(L_T)]$.

In step two we choose a *contagion-free* risk-neutral measure $\widetilde{\mathbf{Q}}$ such that L has $\widetilde{\mathbf{Q}}$-local characteristics $(\lambda^{L,\widetilde{\mathbf{Q}}}, \nu^{\widetilde{\mathbf{Q}}})$ and such that (i) $\widetilde{\mathbf{Q}}$ is equivalent to $\mathbf{P}$ and (ii) the *contagion-free market value* $\mathbb{E}^{\widetilde{\mathbf{Q}}}[e^{-rT}\phi(L_T)]$ coincides with the actuarial value of the contract. By general change-of-measure results for marked point processes (see. e.g., [7, Theorem VIII.T10]) condition (i) is satisfied if $\nu^{\widetilde{\mathbf{Q}}}$ is equivalent to $\nu^{\mathbf{P}}$ and if the Radon Nikodym derivative $\mathrm{d}\nu^{\widetilde{\mathbf{Q}}}/\mathrm{d}\nu^{\mathbf{P}}(z) =: \psi(z)$ and the ratio $\lambda_t^{L,\widetilde{\mathbf{Q}}}/\lambda_t^{L,\mathbf{P}} =: \kappa_t$ satisfy mild integrability conditions; condition (ii) can be ensured by an appropriate choice of parameters. Given the large amount of freedom in choosing $\nu^{\widetilde{\mathbf{Q}}}$ and $\lambda^{L,\widetilde{\mathbf{Q}}}$, we propose to preserve the mathematical structure of the local characteristics of L in the transition from $\mathbf{P}$ to $\widetilde{\mathbf{Q}}$. More precisely, we assume that $\nu^{\widetilde{\mathbf{Q}}}$ belongs to the same class of distributions as $\nu^{\mathbf{P}}$; that W^1 is also a $\widetilde{\mathbf{Q}}$-Brownian motion; and finally that under $\widetilde{\mathbf{Q}}$ the loss intensity is of the form $\lambda_t^{L,\widetilde{\mathbf{Q}}} = c\lambda^{L,\mathbf{P}}(\widetilde{X}_t)\,, \quad 0 \le t \le T$, for some constant $c > 0$, so that under $\widetilde{\mathbf{Q}}$ there is no pricing contagion.[2] The parameter c in the $\widetilde{\mathbf{Q}}$-loss intensity is calibrated to ensure that the contagion free market value of the reinsurance contract equals the actuarial value (which is contagion free by design). Moreover, to account for risk aversion on the part of the reinsurer, the parameters of the claim size distribution can be altered so that large claims are more likely under $\widetilde{\mathbf{Q}}$ than under $\mathbf{P}$.

In step three we model the loss intensity λ^L and the claim size distribution ν under the risk-neutral measure $\mathbf{Q}$. In order to incorporate pricing contagion and the risk of default of R we assume that the risk-neutral loss intensity is of the form $\lambda_t^L = c\lambda^{L,\mathbf{P}}(X_t)$, where X solves the SDE

$$\mathrm{d}X_t = \gamma^X(X_{t-})\mathrm{d}H_t^R + b^X(X_t)\mathrm{d}t + \sigma^X(X_t)\mathrm{d}W_t^1,$$

[2] Note that the change of measure is accomplished via the Radon Nikodym derivative $\frac{\mathrm{d}\widetilde{\mathbf{Q}}}{\mathrm{d}\mathbf{P}}\,|_{\mathcal{G}_T} = \zeta_T$ where ζ solves the SDE

$$\mathrm{d}\zeta_t = \zeta_{t-}(\kappa_t\psi(z) - 1)(m^L(\mathrm{d}t,\mathrm{d}z) - \nu^{\mathbf{P}}(\mathrm{d}z)\lambda_t^{\mathbf{P}}\mathrm{d}t), \quad \zeta_0 = 1;$$

see [7, Theorem VIII.T10] for details. This change of measure affects only the local characteristics of L, the law of the Brownian motions and of the default process stay unchanged.

for $\gamma^X(x) > 0$ and a $\mathbf{Q}$-Brownian motion W^1. Note that the intensity-factor X exhibits an upward jump at the default time τ_R which increases the risk neutral loss intensity, so that under $\mathbf{Q}$ there is pricing contagion. On the other hand there is no need to alter the claim size distribution in the transition from $\widetilde{\mathbf{Q}}$ to $\mathbf{Q}$, that is we take $\nu = \nu^{\widetilde{\mathbf{Q}}}$.

The final task in model calibration is to determine γ^X and the intensity correlation ρ. Here we propose to rely on the expert judgement from experienced underwriters.

REMARK 2.2. *If one lacks sufficient data to calibrate a full-fledged diffusion model for $\lambda^{L,\mathbf{P}}$ or if past loss data warrant a simpler model for the loss intensity one could assume that the $\mathbf{P}$ loss intensity is constant, that is $\lambda_t^{L,\mathbf{P}} = \lambda_0^{L,\mathbf{P}}$; the contagion-free loss intensity is then constant as well, and to account for pricing contagion the $\mathbf{Q}$ loss intensity takes the form $\lambda_t^{L,\mathbf{Q}} = \lambda_0^{L,\mathbf{Q}}(1 + \gamma H_t^R)$ for some $\gamma > 0$. Such a model might be sufficient for certain applications.*

2.3. Model construction. The goal of this section is to provide a step-by-step construction of the model introduced in Section 2.1. Moreover, we establish certain mathematical properties that are needed for the characterization of the credit value adjustment. We start by fixing a filtered probability space $(\Omega, \mathcal{G}, \mathbf{Q})$. Let $W = (W_t)_{t\geq 0}$ be a two-dimensional Brownian motion with components $(W_t^1, W_t^2)_{t\geq 0}$, let $\eta = (\eta_t)_{t\geq 0}$ be a standard Poisson process independent of W, and $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables with identical distribution $\nu(\mathrm{d}z)$, and that are also independent of W and η. Define the process $M = (M_t)_{t\geq 0}$ with $M_t = \sum_{n=1}^{\eta_t} Z_n$. This is a compound Poisson process with intensity equal to one and jump size distribution $\nu(\mathrm{d}z)$. Let the process Y be the unique solution of the SDE

$$\mathrm{d}Y_t = b^Y(Y_t)\mathrm{d}t + \sigma^Y(Y_t)(\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\mathrm{d}W_t^2), \quad Y_0 = y_0 \in \mathbb{R}.$$

We assume that there exists a $\mathcal{G}$-measurable random variable ϑ with unit exponential law, independent of W and M and we define τ_R as

$$\tau_R := \inf\Big\{t \geq 0 : \int_0^t \lambda^R(Y_s)\mathrm{d}s \geq \vartheta\Big\}.$$

By construction the random time τ_R is doubly stochastic with respect to the filtration $\mathbb{F}^W \vee \mathbb{F}^M$ with hazard rate $(\lambda^R(Y_t))_{t\geq 0}$, that is we have for every $t > 0$

$$\mathbf{Q}(\tau_R > t \mid \mathcal{F}_\infty^W \vee \mathcal{F}_\infty^M) = \mathbf{Q}\Big(\int_0^t \lambda^R(Y_s)\mathrm{d}s \leq \vartheta \mid \mathcal{F}_\infty^W \vee \mathcal{F}_\infty^M\Big) = e^{-\int_0^t \lambda^R(Y_s)\mathrm{d}s};$$

see, e.g. [4, Section 8.2.1] or [25, Section 10.5] for details. We define $H_t^R = \mathbf{1}_{\{\tau_R \leq t\}}$, $t \geq 0$, and we introduce the process X as the unique solution to the SDE

$$\mathrm{d}X_t = \gamma^X(X_{t-})\, \mathrm{d}H_t^R + b^X(X_t)\mathrm{d}t + \sigma^X(X_t)\mathrm{d}W_t^1, \quad X_0 = x_0 \in \mathbb{R}.$$

To construct the aggregate claims process we use a time change argument. Define the process $\theta = (\theta_t)_{t\geq 0}$ by $\theta_t := \int_0^t \lambda^L(X_{s-})\mathrm{d}s$ for every $t \geq 0$ and let $N_t := \eta_{\theta_t}$, $t \geq 0$. It is easy to see that $N = (N_t)_{t\geq 0}$ is a doubly stochastic point process with intensity $(\lambda^L(X_t))_{t\geq 0}$ (see, e.g. [22]) and that the loss process is given by $L_t = M_{\theta_t} = \sum_{n=1}^{N_t} Z_n$. Finally we define the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ by

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^L \vee \mathcal{H}_t, \quad t \geq 0,$$

completed with $\mathbf{Q}$-null sets, where $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ is the natural filtration of the process H^R. Notice that the random variables Z_n are $\mathcal{G}_{T_n}$-measurable, with $\{T_n\}_{n\in\mathbb{N}}$ being the sequence of jump times of N. Moreover, τ_R is a stopping time with respect to the filtration $\mathbb{G}$. We also have that M^R in equation (1) is $(\mathbb{G}, \mathbf{Q})$-martingale. This is a consequence of the fact that M^R is a martingale with respect to the filtration $\mathbb{F}^W \vee \mathbb{H}$ and, due to independence between M and W, this is also a martingale with respect to filtration $\mathbb{F}^W \vee \mathbb{H} \vee \mathcal{F}^M_\infty$. Now, since $\mathcal{F}^W_t \vee \mathcal{F}^L_t \vee \mathcal{H}_t \subset \mathcal{F}^W_t \vee \mathcal{F}^M_\infty \vee \mathcal{H}_t$ for every $t \geq 0$, then we have that the martingale property for M^R holds for the filtration $\mathbb{F}^W \vee \mathbb{F}^L \vee \mathbb{H}$.

The contagion-free market. In the remaining part of this section we introduce the *contagion-free* setting which will be used in the computations of the credit value adjustment and of the hedging strategies. Let $\widetilde{X} = (\widetilde{X}_t)_{t\geq 0}$ be the unique solution to the SDE

$$\mathrm{d}\widetilde{X}_t = b^X(\widetilde{X}_t)\mathrm{d}t + \sigma^X(\widetilde{X}_t)\mathrm{d}W^1_t, \quad \widetilde{X}_0 = x_0 \in \mathbb{R}.$$

It is easy to see that $\widetilde{X}$ has the same dynamics as the "original" factor X except for the jump at τ_R. We define $\widetilde{N}_t := \eta_{\widetilde{\theta}_t}$ for every $t \geq 0$, where $\widetilde{\theta}_t = \int_0^t \lambda^L(\widetilde{X}_s)\mathrm{d}s$, then $\widetilde{N} = (\widetilde{N}_t)_{t\geq 0}$ is a doubly stochastic point process with intensity $(\lambda^L(\widetilde{X}_t))_{t\geq 0}$. We can use these processes to construct $\widetilde{L} = (\widetilde{L}_t)_{t\geq 0}$ as follows,

$$\widetilde{L}_t = M_{\widetilde{\theta}_t}, \quad t \geq 0.$$

Notice that before default, the triples (X, N, L) and $(\widetilde{X}, \widetilde{N}, \widetilde{L})$ coincide, that is the processes $(1 - H^R_t)(X_t, N_t, L_t)$ and $(1 - H^R_t)(\widetilde{X}_t, \widetilde{N}_t, \widetilde{L}_t)$ are indistinguishable. We let $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ with

$$\mathcal{F}_t = \mathcal{F}^W_t \vee \mathcal{F}^{\widetilde{L}}_t. \tag{4}$$

The following result holds.

LEMMA 2.3. *The random time τ_R is doubly stochastic with respect to the background filtration $\mathbb{F}$.*

Proof. By the construction of τ_R we have $\mathbf{Q}(\tau_R > t \mid \mathcal{F}^W_\infty \vee \mathcal{F}^M_\infty) = e^{-\int_0^t \lambda^R(Y_s)\mathrm{d}s}$ for every $t \geq 0$. Now we observe that $\lambda^R(Y)$ is adapted to $\mathbb{F}^W$ and so is $(e^{-\int_0^t \lambda^R(Y_s)\mathrm{d}s})_{t\geq 0}$. Moreover we have that

$$\mathcal{F}^W_\infty \vee \mathcal{F}^{\widetilde{L}}_\infty \subseteq \mathcal{F}^W_\infty \vee \mathcal{F}^M_\infty\,,$$

which implies that $\mathbf{Q}(\tau_R > t \mid \mathcal{F}^W_\infty \vee \mathcal{F}^{\widetilde{L}}_\infty) = e^{-\int_0^t \lambda^R(Y_s)\mathrm{d}s}$. ∎

3. Credit Value Adjustment.

To resume the problem, we consider a reinsurance contract between I and R with maturity T and payoff $\phi(L_T)$ for a nonnegative and increasing function ϕ. For technical reasons we assume that ϕ is bounded and Lipschitz continuous; this assumption holds for the examples considered in Section 2.1. Moreover, the counterparty-risk free market value of this contract is given by

$$V^\phi_t := \mathbb{E}^{\mathbf{Q}}\left[e^{-r(T-t)}\phi(L_T)|\mathcal{G}_t\right], \quad 0 \leq t \leq T.$$

We assume that the premium for the contract has been paid at $t = 0$ so that I has no financial obligation towards R. If R defaults before the maturity date T, the insurance company needs to renew her protection, that is she needs to buy a new reinsurance contract at the post-default market value $V^{\phi}_{\tau_R}$. We assume that I receives a recovery payment of size $(1-\delta^R)V^{\phi}_{\tau_R}$ where $\delta^R \in (0,1]$ is the *loss given default* (LGD) of R. Hence I suffers a loss of size $\delta^R V^{\phi}_{\tau_R}$. We denote by $\mathrm{CL} = (\mathrm{CL}_t)_{0\le t\le T}$ the payment stream arising from the counterparty-risk loss. We have that

$$\text{(5)} \qquad \mathrm{CL}_t := \delta^R V^{\phi}_{\tau_R} \mathbf{1}_{\{\tau_R \le t\}} = \delta^R \int_0^t V^{\phi}_s \mathrm{d}H^R_s, \quad 0 \le t \le T.$$

Note that under wrong-way risk, i.e. with $\mathbb{E}^{\mathbf{Q}}\left[V^{\phi}_t | \tau_R = t\right] > \mathbb{E}^{\mathbf{Q}}\left[V^{\phi}_t\right]$, the loss of I at τ_R is higher than its unconditional value. This is an important issue in the management of RCCR. For instance, in the Solvency II regulation it is stated that "As the failure of the counterparty is more likely when the potential loss is high, the LGD (in our case the loss caused by the default of R) should be determined for the case of a stressed situation," see [13]. It is a strong point of our approach that wrong-way risk is generated endogenously by the model. In contrast, in the standard formula of Solvency II ad-hoc adjustments are necessary to account for wrong-way risk.

We define the *credit value adjustment* (CVA) for the reinsurance contract as the market consistent value of the future credit loss, that is

$$\text{(6)} \qquad \mathrm{CVA}_t = \mathbb{E}^{\mathbf{Q}}\left[\int_t^T \delta^R V^{\phi}_s e^{-r(s-t)} \mathrm{d}H^R_s | \mathcal{G}_t\right], \quad 0 \le t \le T.$$

The amount CVA_t can be viewed as a risk reserve that the insurance company has to set aside at time t to cover for losses due to reinsurance counterparty risk. Alternatively, CVA_{t_0} can be viewed as the *pricing adjustment* to account for RCCR at time t_0, that is on $\{\tau_R > t_0\}$ the market consistent value of the cash-flow that is actually received by I is equal to $V^{\phi}_{t_0} - \mathrm{CVA}_{t_0}$. This follows from the following lemma.

Lemma 3.1. *For $0 \le t_0 \le T$ one has*

$$\mathbb{E}^{\mathbf{Q}}\left[\int_{t_0}^T e^{-r(s-t_0)} V^{\phi}_s \mathrm{d}H^R_s \mid \mathcal{G}_{t_0}\right] = \mathbf{1}_{\{\tau_R > t_0\}} \mathbb{E}^{\mathbf{Q}}\left[H^R_T e^{-r(T-t_0)} \phi(L_T) \mid \mathcal{G}_{t_0}\right].$$

Proof. Define the stopping time $\sigma_R := (\tau_R \wedge T) \vee t_0$. Since $(e^{-rt} V^{\phi}_t)_{0\le t\le T}$ is a $(\mathbb{G}, \mathbf{Q})$-martingale and $\sigma_R \le T$, we get from the optional sampling theorem that

$$\text{(7)} \qquad V^{\phi}_{\sigma_R} = \mathbb{E}^{\mathbf{Q}}\left[e^{-r(T-\sigma_R)} \phi(L_T) \mid \mathcal{G}_{\sigma_R}\right].$$

Notice that $\sigma_R = \tau_R$ on the set $\{t_0 < \tau_R \le T\}$ and therefore using equation (7) we get

$$\begin{aligned}
\mathbb{E}^{\mathbf{Q}}\left[\int_{t_0}^T e^{-r(s-t_0)} V^{\phi}_s \mathrm{d}H^R_s \mid \mathcal{G}_{t_0}\right] &= \mathbb{E}^{\mathbf{Q}}\left[\mathbf{1}_{\{t_0 < \tau_R \le T\}} e^{-r(\tau_R - t_0)} V^{\phi}_{\tau_R} \mid \mathcal{G}_{t_0}\right] \\
&= \mathbb{E}^{\mathbf{Q}}\left[\mathbf{1}_{\{t_0 < \tau_R \le T\}} e^{-r(\sigma_R - t_0)} V^{\phi}_{\sigma_R} \mid \mathcal{G}_{t_0}\right] \\
&= \mathbb{E}^{\mathbf{Q}}\left[\mathbb{E}^{\mathbf{Q}}\left[\mathbf{1}_{\{t_0 < \tau_R \le T\}} e^{-r(T-t_0)} \phi(L_T) \mid \mathcal{G}_{\sigma_R}\right] \mid \mathcal{G}_{t_0}\right],
\end{aligned}$$

so that the lemma follows from iterated conditional expectations (as $\mathcal{G}_{t_0} \subseteq \mathcal{G}_{\sigma_R}$). ∎

Now we return to the interpretation of the CVA. Fix $t_0 \in [0,T]$. On $\{\tau_R > t_0\}$ the cash flow actually received by I is given by $\phi(L_T)(1-H_T^R) + (1-\delta^R)\int_{t_0}^T V_s^\phi \mathrm{d}H_s^R$. The expected discounted value of this cash-flow equals

$$V_{t_0}^\phi - \mathbb{E}^{\mathbf{Q}}\left[e^{-r(T-t_0)}\phi(L_T)H_T^R \mid \mathcal{G}_{t_0}\right] + \mathbb{E}^{\mathbf{Q}}\left[\int_{t_0}^T e^{-r(s-t_0)}V_s^\phi \mathrm{d}H_s^R \mid \mathcal{G}_{t_0}\right] - \mathrm{CVA}_{t_0}$$

which is equal to $V_{t_0}^\phi - \mathrm{CVA}_{t_0}$, as the terms in the middle cancel by Lemma 3.1.

Next we want to represent the value of the CVA as classic solution of a partial integro-differential equation (PIDE). This allows for an alternative characterization of the adjusted price in addition to the stochastic representation given in equation (6), and it is essential for the computation of the hedging strategy in Section 4. As a first step we analyze the term $V_{\tau_R}^\phi$ that appears in the definition of the credit loss. Note that the shifted process $(X_{\tau_R+t}, L_{\tau_R+t})_{t\geq 0}$ has the same dynamics as the contagion-free processes $(\widetilde{X}_t, \widetilde{L}_t)_{t\geq 0}$; hence it is a two-dimensional Markov process with generator

$$\begin{aligned}\mathcal{L}^{(\widetilde{L},\widetilde{X})}f(t,l,x) &= \frac{\partial f}{\partial x}(t,l,x)b^X(x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,l,x)(\sigma^X(x))^2 \\ &\quad + \int_{\mathbb{R}^+}\big(f(t,l+z,x) - f(t,l,x)\big)\lambda^L(x)\nu(\mathrm{d}z).\end{aligned} \tag{8}$$

This suggests that $V_{\tau_R}^\phi$ can be described as the solution of a backward equation involving the generator $\mathcal{L}^{(\widetilde{L},\widetilde{X})}$. The next proposition shows that this is in fact correct.

PROPOSITION 3.2. *Under Assumption 2.1, there exists a unique bounded classical solution v^ϕ (i.e. continuous, $\mathcal{C}^1$ in t and $\mathcal{C}^2$ in x) of the following backward PIDE*

$$\frac{\partial v^\phi}{\partial t}(t,l,x) + \mathcal{L}^{(\widetilde{L},\widetilde{X})}v^\phi(t,l,x) = rv^\phi(t,l,x), \quad (t,l,x) \in [0,T)\times\mathbb{R}^+\times\mathbb{R}, \tag{9}$$

with terminal condition $v^\phi(T,l,x) = \phi(l)$. Moreover, it holds for $\tau_R \leq T$ that

$$V_{\tau_R}^\phi = v^\phi\big(\tau_R, \widetilde{L}_{\tau_R}, \widetilde{X}_{\tau_R} + \gamma^X(\widetilde{X}_{\tau_R})\big).$$

Proof. The process $(\widetilde{L},\widetilde{X})$ is a two-dimensional Markov process with pure jump component $\widetilde{L}$ and generator $\mathcal{L}^{(\widetilde{L},\widetilde{X})}$ given in (8). The existence of a classical solution v^ϕ to the backward equation (9) follows from [14]. Moreover, it holds that

$$v^\phi(t,l,x) = \mathbb{E}^{\mathbf{Q}}\left[e^{-r(T-t)}\phi(\widetilde{L}_T) \mid \widetilde{L}_t = l, \widetilde{X}_t = x\right].$$

The strong Markov property thus gives that on $\{\tau_R \leq T\}$,

$$V_{\tau_R}^\phi = v^\phi\big(\tau_R, L_{\tau_R}, X_{\tau_R}\big) = v^\phi(\tau_R, \widetilde{L}_{\tau_R}, \widetilde{X}_{\tau_R} + \gamma^X(\widetilde{X}_{\tau_R})),$$

where in the last equality we used that $L_{\tau_R} = \widetilde{L}_{\tau_R}$, $X_{\tau_R} = \widetilde{X}_{\tau_R} + \gamma^X(\widetilde{X}_{\tau_R})$ and $\widetilde{X}_{\tau_R -} = \widetilde{X}_{\tau_R}$. ∎

Note that the regularity properties of the function v^ϕ ($\mathcal{C}^1$ in t, $\mathcal{C}^2$ in x but only continuous in l) are due to the fact that $\widetilde{L}$ is a pure jump process and therefore the smoothing effect coming from the diffusion does not apply in the l direction. In the statement of Proposition 3.2 we refer for brevity to Assumption 2.1. However, Proposition 3.2 does not involve the process Y and therefore some of the conditions in the list (A1)–(A4) are unnecessary.

PROPOSITION 3.3. *Under Assumptions 2.1 the value of the* CVA *is given by*

$$\mathrm{CVA}_t = \delta^R(1-H_t^R) f^{\mathrm{CVA}}(t, L_t, X_t, Y_t) \tag{10}$$

where $f^{\mathrm{CVA}} : [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is a classical solution (i.e. continuous, $\mathcal{C}^1$ in t and $\mathcal{C}^2$ in (x,y)) of the following backward PIDE

$$\frac{\partial f^{\mathrm{CVA}}}{\partial t} + \mathcal{L}^{(\widetilde{L},\widetilde{X},Y)} f^{\mathrm{CVA}} + \lambda^R(y) v^\phi(t,l,x+\gamma^X(x)) = (\lambda^R(y)+r) f^{\mathrm{CVA}}, \tag{11}$$

for all $(t,l,x,y) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R}^2$ with terminal condition $f^{\mathrm{CVA}}(T,l,x,y) = 0$. The operator $\mathcal{L}^{(\widetilde{L},\widetilde{X},Y)}$ (the generator of the three-dimensional Markov process $(\widetilde{L},\widetilde{X},Y)$) is given by

$$\begin{aligned}\mathcal{L}^{(\widetilde{L},\widetilde{X},Y)} f &= \frac{\partial f}{\partial x} b^X(x) + \frac{\partial f}{\partial y} b^Y(y) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\sigma^X(x))^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\sigma^Y(y))^2 \\ &\quad + \frac{\partial^2 f}{\partial x \partial y}\rho\sigma^X(x)\sigma^Y(y) + \int_{\mathbb{R}^+} (f(t,l+z,x,y) - f(t,l,x,y))\lambda^L(x)\nu(\mathrm{d}z),\end{aligned} \tag{12}$$

where f is always evaluated at (t,l,x,y).

Proof. The CL is a so-called payment-at-default claim (see for instance [25, Section 10.5]). Proposition 3.2 allows to express its payoff at τ_R in terms of contagion free quantities. Then we get that

$$\mathrm{CVA}_t = \mathbb{E}^{\mathbf{Q}}\left[\int_t^T \delta^R v^\phi(s, \widetilde{L}_s, \widetilde{X}_s + \gamma^X(\widetilde{X}_s)) e^{-r(s-t)} \mathrm{d}H_s^R \mid \mathcal{G}_t\right]. \tag{13}$$

In equation (13) we can replace $\mathcal{G}_t$ with $\mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{F}_t$ is defined in (4), since these sigma fields coincide up to time τ_R. Then we get from Lemma 2.3 and [25, Theorem 10.19] that

(14)

$$\mathrm{CVA}_t = \delta^R(1-H_t^R)\mathbb{E}^{\mathbf{Q}}\left[\int_t^T v^\phi(s, \widetilde{L}_s, \widetilde{X}_s + \gamma^X(\widetilde{X}_s))\lambda^R(Y_s) e^{-\int_t^s (r+\lambda^R(Y_u))\mathrm{d}u} \mathrm{d}s \mid \mathcal{F}_t\right].$$

Note that the process $(\widetilde{L},\widetilde{X},Y)$ is Markovian with respect to the filtration $\mathbb{F}$ with generator $\mathcal{L}^{(\widetilde{L},\widetilde{X},Y)}$ as in (12). It follows that there is a function $f^{\mathrm{CVA}} : [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ such that

$$\mathrm{CVA}_t = \delta^R(1-H_t^R) f^{\mathrm{CVA}}(t, \widetilde{L}_t, \widetilde{X}_t, Y_t).$$

Then, by applying [14, Theorem 2.4] we get that f^{CVA} is a classical solution of the backward PIDE (11). Finally note that on the event $\{\tau_R > t\}$, $1 - H_t^R = 1$ and also $\widetilde{L}_t = L_t$, $\widetilde{X}_t = X_t$, which implies (10). ∎

EXAMPLE 3.4. *In the numerical analysis we consider a special case of our setting. There the loss intensity λ^L is constant except for an upward jump at time τ_R that models price contagion. In this case we may identify the intensity λ^L and the intensity-factor process X (i.e. $\lambda^L(\cdot)$ is the identity function) and assume that*

$$\lambda^L(X_t) = X_t = x_0(1 + H_t^R\gamma), \quad 0 \le t \le T, \tag{15}$$

for constants $x_0 > 0$ and $\gamma > 0$. Here the parameter γ models the percentage change in the loss intensity at τ_R. We now calculate the credit value adjustment for this situation. Under (15) the process $\widetilde{L}$ is a compound Poisson process with intensity x_0, jump-size distribution $\nu(\mathrm{d}z)$ and generator

$$\mathcal{L}^{\widetilde{L}}_{x_0} f(t,l) = x_0 \int_{\mathbb{R}^+} \big(f(t,l+z) - f(t,l)\big)\nu(\mathrm{d}z).$$

For given $x_0 > 0$, define the function $(t,l) \mapsto v^{\phi}(x_0;t,l)$ as the solution of the backward integral equation

$$\frac{\partial v^{\phi}}{\partial t}(x_0;t,l) + \mathcal{L}^{\widetilde{L}} v^{\phi}(x_0;t,l) = r v^{\phi}(x_0;t,l), \quad (t,l) \in [0,T) \times \mathbb{R}^+,$$

with terminal condition $v^{\phi}(x_0;T,l) = \phi(l)$. Then, the post default value of the reinsurance contract is given by[3]

$$V^{\phi}_{\tau_R} = v^{\phi}(x_0(1+\gamma); \tau_R, \widetilde{L}_{\tau_R}).$$

With this we get that credit value adjustment satisfies $\mathrm{CVA}_t = \delta^R(1 - H^R_t) f^{\mathrm{CVA}}(x_0;t,\widetilde{L}_t,Y_t)$, where the function $(t,l,y) \mapsto f^{\mathrm{CVA}}(x_0;t,l,y)$ is the solution of the backward PIDE

$$\begin{aligned} (16) \qquad \frac{\partial f^{\mathrm{CVA}}}{\partial t}(x_0;t,l,y) + \mathcal{L}^{(\widetilde{L},Y)}_{x_0} f^{\mathrm{CVA}}(x_0;t,l,y) + \lambda^R(y) v^{\phi}(x_0(1+\gamma);t,l) \\ = (\lambda^R(y) + r) f^{\mathrm{CVA}}(x_0;t,l,y), \end{aligned}$$

for every $(t,l,y) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R}$ with terminal condition $f^{\mathrm{CVA}}(x_0;T,l,y) = 0$, and where for a generic continuous function $f(l,y)$ which is $\mathcal{C}^2$ in y, the operator $\mathcal{L}^{(\widetilde{L},Y)}_{x_0}$ is given by

$$\mathcal{L}^{(\widetilde{L},Y)}_{x_0} f(l,y) = \frac{\partial f}{\partial y}(l,y) b^Y(y) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(l,y)(\sigma^Y(y))^2 + x_0 \int_{\mathbb{R}^+} (f(l+z,y) - f(l,y))\nu(\mathrm{d}z).$$

Note that in this example the variable corresponding to loss intensity drops out of the equation (16) and therefore (A2) in Assumption 2.1 can be replaced by the simpler condition

(A2') There is some $\beta > 0$ such that $\sigma^Y(\cdot) > \beta$.

4. Hedging of Reinsurance Counterparty Credit Risk. In this section we investigate how the insurance company can reduce the losses arising from the default of the reinsurer by a dynamically adjusted position in a credit default swap (CDS) on R. A CDS is a natural hedging instrument for credit risk since it makes a payment at τ_R, that is exactly when the counterparty risk loss arises. Moreover, there is a reasonably liquid market for CDSs on major reinsurane companies. Another option for managing counterparty risk would be a dynamically adjusted collateralization strategy as in [20]; however, one of the advantages of hedging with CDS contracts is that a strategy can be implemented unilaterally by I. In our setting there are several sources of randomness that do not correspond to traded assets, such as the loss process L or the loss intensity λ^L, and therefore perfect hedging is not possible. To deal with

[3] Of course one could use other actuarial techniques such as Panjer recursion to compute v^{ϕ}.

the ensuing market incompleteness we resort to a quadratic hedging method. Precisely we will consider self financing strategies and minimize the quadratic hedging error at the maturity date.

To proceed with a formal analysis of the hedging problem we need to discuss the dynamics of a self-financing CDS trading strategy. This issue is taken up next.

4.1. Dynamics of a CDS trading strategy. We consider a CDS contract on R with fixed running spread premium $\zeta > 0$ and with default payment given by the deterministic loss given default $\delta^{\mathrm{CDS}} \in (0,1]$ of R. To simplify the exposition we assume that the premium payments are made continuously. The cashflow stream associated to the CDS (from the viewpoint of I) is therefore given by

$$D_t^R = \delta^{\mathrm{CDS}} H_t^R - \zeta \int_0^t (1 - H_u^R)\mathrm{d}u, \quad 0 \le t \le T, \tag{17}$$

where the first term refers to the payment at default and the second term is the premium payment. Note that (17) describes the cash-flows of a CDS contract with notional equal to one; holding m units of this contract is the same as holding one CDS contract with notional m.

The present value of the future payments of the CDS is given by

$$\begin{aligned}\Lambda_t &:= \mathbb{E}^{\mathbf{Q}}\left[\int_t^T e^{-r(u-t)}\mathrm{d}D_u^R | \mathcal{G}_t\right]\\ &= \mathbb{E}^{\mathbf{Q}}\left[\delta^{\mathrm{CDS}} \int_t^T e^{-r(u-t)}\mathrm{d}H_u^R - \zeta \int_t^T e^{-r(u-t)}(1 - H_u^R)\mathrm{d}u | \mathcal{G}_t\right].\end{aligned}$$

Similarly as in Section 3, we characterize the process Λ in terms of the classical solution of a backward partial differential equation (PDE).

Proposition 4.1. *Under Assumptions 2.1 the process Λ is given by*

$$\Lambda_t = (1 - H_t^R) g(t, Y_t)$$

where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a classical solution (i.e. $\mathcal{C}^1$ in t and $\mathcal{C}^2$ in y) of the following backward PDE

$$\frac{\partial g}{\partial t}(t,y) + \mathcal{L}^Y g(t,y) + (\delta^{\mathrm{CDS}}\lambda^R(y) - \zeta) = (\lambda^R(y) + r) g(t,y), \quad (t,y) \in [0,T) \times \mathbb{R}, \tag{18}$$

with terminal condition $g(T,y) = 0$. Here the operator $\mathcal{L}^Y$ is the generator of Y, that is

$$\mathcal{L}^Y f(y) = \frac{\partial f}{\partial y}(y) b^Y(y) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(y)(\sigma^Y(y))^2. \tag{19}$$

Proof. Since M^R in (1) is a $\mathbb{G}$- martingale we have that

$$\Lambda_t = \mathbb{E}^{\mathbf{Q}}\left[\int_t^T e^{-r(u-t)}(\delta^{\mathrm{CDS}}\lambda^R(Y_u) - \zeta)(1 - H_u^R)\mathrm{d}u | \mathcal{G}_t\right] \tag{20}$$

Using Fubini's theorem, Lemma 2.3 and [25, Theorem 10.19] we get that the right hand side of (20) is equal to

$$(1 - H_t^R)\mathbb{E}^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^u (r+\lambda^R(Y_s))\mathrm{d}s}(\delta^{\mathrm{CDS}}\lambda^R(Y_u) - \zeta)\mathrm{d}u|\mathcal{F}_t\right] \tag{21}$$

By Markovianity of the process Y with respect to filtration $\mathbb{F}$, there exists a function g such that conditional expectation in (21) is equal to $g(t, Y_t)$. Denote by $\mathcal{L}^Y$ the generator of Y given by (19). Then it is easily seen that under Assumption 2.1, g is the classical solution of (18), see, e.g. [27, Theorem 8.2.1]. □

Finally we define the *discounted gains process* of the CDS (the past cashflows and the present value of the future cashflows, both discounted back to time zero) by

$$S_t = e^{-rt}\Lambda_t + \int_0^t e^{-ru}\mathrm{d}D_u^R, \quad 0 \le t \le T. \tag{22}$$

Note that $S_t = \mathbb{E}^{\mathbf{Q}}\left[\int_0^T e^{-ru}\mathrm{d}D_u^R|\mathcal{G}_t\right]$ for every $0 \le t \le T$. Therefore S is a square integrable $(\mathbb{G}, \mathbf{Q})$-martingale (S is even bounded as the cash flow stream D^R is bounded). Consider now a self-financing trading strategy $\xi = (\xi^0, \xi^1)$, where ξ_t^1 is the notional of the CDS position at time t and where ξ_t^0 is the cash position at time t. Then the value of this strategy at time $0 \le t \le T$ equals $V_t(\xi) = \xi_t^1\Lambda_t + \xi_t^0 e^{-rt}$, and the strategy is *self-financing* if the discounted value $\widetilde{V}_t(\xi) = e^{-rt}V_t(\xi)$ satisfies

$$\widetilde{V}_t(\xi) = V_0(\xi) + \int_0^t \xi_s^1 \mathrm{d}S_s\,, \quad 0 \le t \le T\,.$$

4.2. Quadratic hedging. Next we formalize the quadratic criterion that is used to determine the optimal hedging strategy. We call a self-financing trading strategy $\xi = (\xi^0, \xi^1)$ *admissible* if ξ^0 is $\mathbb{G}$-adapted and ξ^1 is $\mathbb{G}$-predictable and satisfies the integrability condition

$$\mathbb{E}^{\mathbf{Q}}\left[\int_0^T (\xi_u^1)^2 \mathrm{d}\langle S\rangle_u\right] < \infty\,. \tag{23}$$

Here $\langle S\rangle$ denotes the *predictable quadratic variation* of the martingale S (the predictable compensator of the pathwise quadratic variation $[S]$ of S). Condition (23) ensures that the discounted value process $V(\xi)$ is a right continuous and square integrable martingale. The hedging problem amounts to finding a self-financing admissible strategy ξ^* with initial value $V_0(\xi^*)$ and CDS position $\xi^{1,*}$ that minimizes the quadratic hedging error

$$\mathbb{E}^{\mathbf{Q}}\left[\left(\int_0^T e^{-rt}\delta^R V_t^{\phi}\mathrm{d}H_t^R - \left(V_0(\xi) + \int_0^T \xi_t^1 \mathrm{d}S_t\right)\right)^2\right]. \tag{24}$$

Such a strategy will be called $\mathbf{Q}$*-mean-variance minimizing.*

Remark 4.2. *We continue with a few comments on the hedging criterion.*

1) Minimizing the quadratic hedging error with respect to the risk-neutral measure $\mathbf{Q}$*, instead of the historical measure* $\mathbf{P}$*, has a couple of advantages. First, the ensuing*

CDS position $\xi^{1,*}$ is time-consistent: the CDS strategy that minimizes the conditional quadratic hedging error

$$\mathbb{E}^{\mathbf{Q}}\left[\left(\int_t^T e^{-rs}\delta^R V_s^{\phi}\mathrm{d}H_s^R - \left(V_t(\xi) + \int_t^T \xi_s^1 \mathrm{d}S_s\right)\right)^2 \,\Big|\, \mathcal{G}_t\right]$$

over the period $[t,T]$ is the restriction of $\xi^{1,*}$ to the interval $[t,T]$. This is in general not true for a $\mathbf{P}$-mean-variance minimizing strategy. Moreover, since the default and loss intensities under $\mathbf{Q}$ are typically higher than the corresponding $\mathbf{P}$-intensities, more mass is put in expectation (24) on states where the counterparty-risk loss is large and the $\mathbf{Q}$-mean-variance minimizing strategy will track the credit loss more closely in those states than a $\mathbf{P}$-mean-variance-minimizing strategy; this adds an additional layer of prudence to our approach. Finally a $\mathbf{Q}$-mean-variance-minimizing strategy is comparatively easy to determine and the solution has a clear economic interpretation.

2) As an alternative to $\mathbf{Q}$-mean-variance minimization one might consider risk minimization *under $\mathbf{Q}$ as hedging criterion. The investment in the risky asset (the CDS in our setting) is the same for both approaches; the only difference is that in the mean-variance-hedging approach a self-financing strategy is followed until time T where the hedging error takes the form of a lump sum adjustment. In the risk minimization approach on the other hand the portfolio value is adjusted continuously at any $0 < t \le T$. Note however that mean-variance hedging and risk minimization lead to different strategies if one works under the historical measure. For an in-depth discussion of these issues we refer to [28].*

To determine the $\mathbf{Q}$-mean-variance minimizing strategy we first introduce the discounted gain process M^{CL} associated with the credit loss. This process is given by

$$M_t^{\mathrm{CL}} = \mathbb{E}^{\mathbf{Q}}\left[\int_0^T e^{-rs}\mathrm{d}\,\mathrm{CL}_s \,|\mathcal{G}_t\right] = \int_0^t e^{-rs}\mathrm{d}\,\mathrm{CL}_s + e^{-rt}\,\mathrm{CVA}_t, \quad 0 \le t \le T, \tag{25}$$

where CL represents the payment stream arising from the counterparty-risk loss, see equation (5). Recall that the payoff ϕ of the reinsurance contract is bounded by assumption. This implies that CL is bounded, so that M^{CL} is a bounded and hence in particular a square integrable $(\mathbb{G},\mathbf{Q})$-martingale. Since the discounted gain process of the CDS in equation (22) is a square integrable $(\mathbb{G},\mathbf{Q})$-martingale, it is well known that the $\mathbf{Q}$-mean-variance optimal strategy can be determined with the help of the Galtchouk-Kunita-Watanabe decomposition of M^{CL} with respect to S. This result ensures the existence of a predictable process $\xi^{1,*}$ satisfying (23) and of a martingale A, null at time zero, which is strongly orthogonal to S (that is the product of the two martingales $(S_tA_t)_{0\le t\le T}$ is also a martingale or, equivalently, the predictable quadratic covariation $\langle S, A\rangle$ vanishes) such that

$$M_t^{\mathrm{CL}} = M_0^{\mathrm{CL}} + \int_0^t \xi_u^{1,*}\mathrm{d}S_u + A_t, \quad \mathbf{Q}-a.s. \quad 0 \le t \le T. \tag{26}$$

Then the strategy ξ^* with CDS position $\xi^{1,*}$ and initial value $V_0(\xi^*) = M_0^{\mathrm{CL}}$ is admissible and $\mathbf{Q}$-mean-variance minimizing. A detailed proof of this result can be found in [28]. Intuitively, decomposition (26) permits to decompose the payment stream CL into its attainable part given by $\int \xi_t^{1,*}\mathrm{d}S_t$, and an unattainable part A corresponding to non-hedgeable risk.

Identifying $\xi^{1,*}$ entails taking the predictable covariation with respect to S on both sides of equation (26). Using orthogonality between A and S, we get that

$$\langle M^{\mathrm{CL}}, S\rangle_t = \int_0^t \xi^{1,*}_u \mathrm{d}\langle S\rangle_u, \quad 0 \le t \le T,$$

where $\langle M^{\mathrm{CL}}, S\rangle$ denotes the predictable quadratic covariation between martingales M^{CL} and S. This implies that $\xi^{1,*}$ can be identified as predictable version of the Radon Nikodym density $\frac{\mathrm{d}\langle M^{\mathrm{CL}},S\rangle}{\mathrm{d}\langle S\rangle}$. Notice that since M^{CL} and S are square integrable martingales, the process $\xi^{1,*}$ obtained via this construction naturally satisfies the integrability condition (23). Computing the density $\frac{\mathrm{d}\langle M^{\mathrm{CL}},S\rangle}{\mathrm{d}\langle S\rangle}$ is the key point in the proof of the following theorem where we determine the **Q**-mean-variance minimizing strategy.

THEOREM 4.3. *The* **Q***-mean-variance minimizing strategy is characterized by the initial value* $V_0(\xi^*) = \mathrm{CVA}_0$ *and by the CDS position* $\xi^{1,*}_t = \frac{\mathrm{d}\langle M^{CL},S\rangle_t/\mathrm{d}t}{\mathrm{d}\langle S\rangle_t/\mathrm{d}t}$*, for every* $0 \le t \le T$*, where*

(27)

$$\begin{aligned}\frac{\mathrm{d}\langle M^{\mathrm{CL}}, S\rangle_t}{\mathrm{d}t} = {} & \delta^R e^{-2rt}(1-H^R_{t-})\Big\{\rho\sigma^X(X_{t-})\sigma^Y(Y_t)\frac{\partial f^{\mathrm{CVA}}}{\partial x}(t, L_{t-}, X_{t-}, Y_t)\frac{\partial g}{\partial y}(t, Y_t)\\ & +(\sigma^Y(Y_t))^2\ \frac{\partial f^{\mathrm{CVA}}}{\partial y}(t, L_{t-}, X_{t-}, Y_t)\frac{\partial g}{\partial y}(t, Y_t)\\ & +\lambda^R(Y_t)\big(\delta^{\mathrm{CDS}} - g(t, Y_t)\big)\big(v^\phi(t, L_{t-}, X_{t-}+\gamma^X(X_{t-})) - f^{\mathrm{CVA}}(t, L_{t-}, X_{t-}, Y_t)\big)\Big\}\end{aligned}$$

and

(28)

$$\frac{\mathrm{d}\langle S\rangle_t}{\mathrm{d}t} = e^{-2rt}(1-H^R_{t-})\Big\{\lambda^R(Y_t)(\delta^{\mathrm{CDS}} - g(t, Y_t))^2 + (\sigma^Y(Y_t))^2\left(\frac{\partial g}{\partial y}(t, Y_t)\right)^2\Big\}.$$

Proof. By definition $M^{\mathrm{CL}}_0 = \mathrm{CVA}_0$ which gives the initial value of the strategy. In order to determine $\xi^{1,*}$ note that in our setting the processes $\langle M^{\mathrm{CL}}, S\rangle$ and $\langle S\rangle$ are absolutely continuous with respect to Lebesgue measure. This implies that **Q**-a.s.

$$\frac{\mathrm{d}\langle M^{\mathrm{CL}}, S\rangle_t}{\mathrm{d}\langle S\rangle_t} = \frac{\mathrm{d}\langle M^{\mathrm{CL}},S\rangle_t/\mathrm{d}t}{\mathrm{d}\langle S\rangle_t/\mathrm{d}t}, \quad 0 \le t \le T.$$

To derive the processes $\frac{\mathrm{d}\langle M^{\mathrm{CL}},S\rangle_s}{\mathrm{d}s}$ and $\frac{\mathrm{d}\langle S\rangle_s}{\mathrm{d}s}$ we compute the pathwise quadratic (co)variations $[M^{\mathrm{CL}}, S]$, respectively $[S]$, and we use that $\langle M^{\mathrm{CL}}, S\rangle$, respectively $\langle S\rangle$, is the predictable compensator of these processes. We recall that M^R is the compensated martingale given in equation (1) and denote by $\widetilde{m}(\mathrm{d}t, dz)$ the compensated jump measure $\widetilde{m}(\mathrm{d}t, dz) = m^L(\mathrm{d}t, \mathrm{d}z) - \lambda^L(X_{t-})\nu(\mathrm{d}z)$. From the PIDE characterization of the CVA in Proposition 3.3 and the Itô formula, see Appendix A for the detailed computations, we get that the martingale M^{CL} in (25) is explicitly given by

$$M_t^{\mathrm{CL}} = M_0^{\mathrm{CL}} + \delta^R \int_0^t e^{-rs}(v^\phi(s, L_{s-}, X_{s-} + \gamma^X(X_{s-})) - f^{\mathrm{CVA}}(s, L_{s-}, X_{s-}, Y_s))\mathrm{d}M_s^R$$

$$+ \delta^R \int_0^t e^{-rs}(1-H_{s-}^R)\sigma^X(X_{s-})\frac{\partial f^{\mathrm{CVA}}}{\partial x}(s, L_{s-}, X_{s-}, Y_s)\mathrm{d}W_s^1$$

$$+ \delta^R \int_0^t e^{-rs}(1-H_{s-}^R)\rho\ \sigma^Y(Y_s)\frac{\partial f^{\mathrm{CVA}}}{\partial y}(s, L_{s-}, X_{s-}, Y_s)\mathrm{d}W_s^1$$

$$+ \delta^R \int_0^t e^{-rs}(1-H_{s-}^R)\sigma^Y(Y_s)\frac{\partial f^{\mathrm{CVA}}}{\partial y}(s, L_{s-}, X_{s-}, Y_s)\sqrt{1-\rho^2}\ \mathrm{d}W_s^2$$

$$+ \delta^R\int_0^t e^{-rs}(1-H_{s-}^R)\int_{\mathbb{R}^+}\big(f^{\mathrm{CVA}}(s, L_{s-}+z, X_{s-}, Y_s) - f^{\mathrm{CVA}}(s, L_{s-}, X_{s-}, Y_s)\big)\widetilde{m}(\mathrm{d}s, \mathrm{d}z),$$

In a similar way we obtain the martingale decomposition of the process S. It holds that for every $0 \le t \le T$,

$$S_t = S_0 + \int_0^t e^{-rs}(\delta^{\mathrm{CDS}} - g(s, Y_s))\mathrm{d}M_s^R$$
$$+ \int_0^t e^{-rs}(1 - H_{s-}^R)\sigma^Y(Y_s)\frac{\partial g}{\partial y}(s, Y_s)\left(\rho\mathrm{d}W_s^1 + \sqrt{1-\rho^2}\mathrm{d}W_s^2\right).$$

Then the quadratic covariation of the two martingales M^{CL} and S and for the quadratic variation of S is

$$\mathrm{d}[M^{\mathrm{CL}}, S]_t = \delta^R e^{-2rt}(1 - H_{t-}^R)\rho\ \sigma^X(X_{t-})\sigma^Y(Y_t)\frac{\partial f^{\mathrm{CVA}}}{\partial x}(t, L_{t-}, X_{t-}, Y_t)\frac{\partial g}{\partial y}(t, Y_t)\mathrm{d}t$$

$$+ \delta^R e^{-2rt}(1 - H_{t-}^R)(\sigma^Y(Y_t))^2\frac{\partial f^{\mathrm{CVA}}}{\partial y}(t, L_{t-}, X_{t-}, Y_t)\frac{\partial g}{\partial y}(t, Y_t)\mathrm{d}t$$

$$+ \delta^R e^{-2rt}\big(\delta^{\mathrm{CDS}} - g(t, Y_t)\big)\big(v^\phi(t, L_{t-}, X_{t-} + \gamma^X(X_{t-})) - f^{\mathrm{CVA}}(t, L_{t-}, X_{t-}, Y_t)\big)\mathrm{d}H_t^R,$$

$$\mathrm{d}[S]_t = e^{-2rt}(\delta^{\mathrm{CDS}} - g(t, Y_t))^2\mathrm{d}H_t^R + e^{-2rt}(1 - H_{t-}^R)(\sigma^Y(Y_t))^2\left(\frac{\partial g}{\partial y}(t, Y_t)\right)^2\mathrm{d}t. \qquad \square$$

The predictable quadratic variation is then obtained by computing predictable compensators, which leads to (27) and (28) and implies the result.

Special cases and interpretation. In order to understand the form of $\xi^{1,*}$ it is instructive to consider first the limiting case where $\sigma^X = \sigma^Y = 0$ and where $\lambda_t^L = X_t = x_0(1 + H_t^R\gamma)$ for some $\gamma > 0$ and $\lambda_t^R = \lambda^R(y_0) > 0$ for every $0 \le t \le T$. In that setting we can consider both x_0 and y_0 as parameters and get that

$$\xi_t^{1,*} = (1 - H_{t-}^R)\frac{\delta^R\big(v^\phi(x_0(1+\gamma); t, L_{t-}) - f^{\mathrm{CVA}}(x_0, y_0; t, L_{t-})\big)}{\delta^{\mathrm{CDS}} - g(t, y_0)}, \quad 0 \le t \le T.$$

It follows that the CDS strategy generates at τ_R a payment of size $\delta^R(v^\phi(x_0(1+\gamma); t, L_{\tau_R}) - f^{\mathrm{CVA}}(x_0, y_0; \tau_R, L_{\tau_R}))$, that is the strategy perfectly compensates the counterparty-risk loss at τ_R (hedging of jump risk). Note however, that the CDS position $\xi_t^{1,*}$ - and hence the premium payments - depends on the random quantity L_t, so that the quadratic hedging error (24) of the strategy is strictly positive.

For $\sigma^Y > 0$ the strategy balances the hedging of jump risk and the hedging against fluctuations in the default intensity factor Y (hedging of spread risk). The optimal mean-variance strategy in the setting of Example 3.4 can be obtained by letting $\sigma^X = 0$. Using the special notation for this case we obtain that

$$\xi_t^{1,*} = (1 - H_{t-}^R) \frac{\delta^R \lambda^R(Y_t)\big(\delta^{\mathrm{CDS}} - g(t,Y_t)\big)\big(v^\phi(x_0(1+\gamma);t,L_{t-}) - f^{\mathrm{CVA}}(x_0;t,L_{t-},Y_t)\big)}{\lambda^R(Y_t)(\delta^{\mathrm{CDS}} - g(t,Y_t))^2 + (\sigma^Y(Y_t))^2 \left(\frac{\partial g}{\partial y}(t,Y_t)\right)^2}$$

$$+ (1 - H_{t-}^R) \frac{\delta^R (\sigma^Y(Y_t))^2 \frac{\partial f^{\mathrm{CVA}}}{\partial y}(x_0;t,L_{t-},Y_t) \frac{\partial g}{\partial y}(t,Y_t)}{\lambda^R(Y_t)(\delta^{\mathrm{CDS}} - g(t,Y_t))^2 + (\sigma^Y(Y_t))^2 \left(\frac{\partial g}{\partial y}(t,Y_t)\right)^2}.$$

If $\sigma^X(\cdot)$, $\sigma^Y(\cdot)$ and ρ are all strictly positive, then an additional cross term $\rho\sigma^X\sigma^Y \frac{\partial f^{\mathrm{CVA}}}{\partial x}\frac{\partial g}{\partial y}$ appears in (27). It is intuitively clear that both partial derivatives are positive[4], so that the CDS position $\xi^{1,*}$ is increased by this term. This is due to the fact that some of the risk caused by fluctuations in the non-traded loss intensity factor X can be hedged by increasing the position in the correlated CDS contract.

5. Numerical Experiments.

In this section we present results from numerical experiments that complement the theoretical analysis. In Section 5.1 we focus on the relative importance of dependence and pricing contagion for wrong way risk; in Section 5.2 we study **Q**-mean-variance-minimizing strategies and we compare their performance to that of a static strategy.

Throughout our analysis we consider the following setup. We identify processes the X, Y and λ^L, λ^R, that is we assume that $\lambda^L(\cdot)$ and $\lambda^R(\cdot)$ are the identity functions. The default intensity follows a CIR process with the dynamics

$$\mathrm{d}Y_t = (0.05 - Y_t)\mathrm{d}t + 0.1\sqrt{Y_t}(\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\mathrm{d}W_t^2), \quad Y_0 = 0.05;$$

this allows for an explicit formula for the price of the CDS, see, e.g. [18]. For the loss intensity we consider a jump diffusion of the form

$$\mathrm{d}X_t = \gamma X_{t-}\, \mathrm{d}H_t^R + \kappa(100 - X_t)\mathrm{d}t + \sigma X_t \mathrm{d}W_t^1, \quad X_0 \in \mathbb{R}^+.$$

If we take $\kappa = \sigma = 0$ we recover the case of Example 3.4 where the loss intensity has a jump at default and is otherwise constant. Finally, we assume that claim sizes are Gamma(α, β) distributed. We consider a reinsurance contract of stop loss type with payoff $\phi(L_T) = [L_T - 90]^+$, capped at 200, we set the interest rate to $r = 0$ and the loss-given-default of R and of the CDS to $\delta^R = \delta^{\mathrm{CDS}} = 1$.

Next we briefly discuss the methods used in the numerical analysis. The main task is to calculate the CVA in (10). Using the equivalent formulation in (14) we see that this amounts to evaluating the expectation

$$\mathbb{E}^{\mathbf{Q}}\left[\int_t^T v^\phi(s, \widetilde{L}_s, \widetilde{X}_s + \gamma\widetilde{X}_s)\, Y_s\, e^{-\int_t^s Y_u \mathrm{d}u}\mathrm{d}s \;\Big|\; \widetilde{L}_t = l, \widetilde{X}_t = x, Y_t = y\right].$$

We evaluate this term using Monte Carlo simulation. In general this is a nested Monte Carlo problem, as one needs also to compute the default free value of the reinsurance contract $v^\phi(t, \widetilde{L}_t, \widetilde{X}_t + \gamma\widetilde{X}_t)$, for every $0 \le t \le T$. For the case where $\kappa = \sigma = 0$,

[4] A higher loss intensity makes a large credit loss more likely, thereby increasing the CVA, and a higher default intensity increases the value of the future CDS payments.

$\widetilde{L}$ follows a compound Poisson process and we may use Panjer recursion. For the general case, we mostly use a regression-based approach to reduce the computational cost (see, [21, Chapter 8.6]). The computation of the mean-variance minimizing hedging strategies involves computing derivatives of the functions f^{CVA} and g. These are computed via a Monte Carlo approach, following [21, Chapter 7.2].

5.1. CVA and wrong-way risk. In this section we analyse the impact of the pricing contagion and the correlation between the loss and the default intensities on the CVA by varying the parameters γ and ρ. We assume that $\sigma = 0.2$ and that claim sizes are Gamma(1,1) distributed.

In Figure 1 we display the CVA at time 0 for different values of $\gamma \in [0, 1]$ (left panel) and for different correlation levels $\rho \in [0, 1]$ (right panel). In these plots we fixed $\kappa = 0.5$. We see that CVA_0 increases in both ρ and γ, which is in line with (3). The effect of price contagion (i.e. variation in γ) is quite pronounced and dominates the effect of dependence between intensities (i.e. variation in ρ), and we conclude that it is very important to incorporate price contagion into the analysis of RCCR.

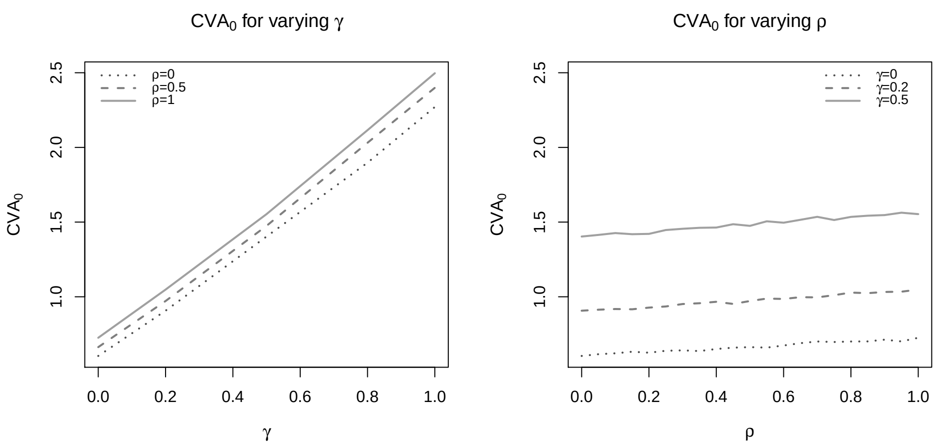

FIG. 1. *Left:* CVA_0 *for varying contagion parameter* γ. *Right:* CVA_0 *for varying correlation* ρ.

5.2. Performance of hedging strategies. We now compute the hedging strategies corresponding to different parameter choices and we compare their performance to that of a static strategy. Precisely we consider the three cases described in Table 1 below. Case 1 and Case 2 correspond to a loss intensity that stays constant with a single jump at time τ_R, where it increases by 20%. The parameters of the claims size distribution and the loss intensity are chosen in such a way that the expected contagion-free loss is the same ($\mathbb{E}^{\mathbf{Q}}\big[\widetilde{L}\big] = 100$). However in Case 1 the insurance company experiences small but frequent losses whereas in Case 2 there are infrequent but large losses. Intuitively we therefore expect hedging to be more difficult in the second case.

In addition to the dynamic $\mathbf{Q}$-mean-variance minimizing strategies from Theorem 4.3 we considered two simpler strategies. First we considered a *static CDS hedging strategy* where the value of the CVA at $t = 0$ is invested in the CDS and where the position is not adjusted over time (in mathematical terms $V_0(\xi) = \mathrm{CVA}_0$ and $\xi_t^1 = \frac{\mathrm{CVA}_0}{\zeta}, 0 \le t \le \tau^R \wedge T$). Moreover we considered a strategy labelled *unhedged*

	X_0	γ	κ	σ	ρ	α	β
Case 1:	100	0.2	0	0	0	1	1
Case 2:	10	0.2	0	0	0	10	1
Case 3:	100	0	1	0.2	0.2	1	1

TABLE 1

Parameters used in the analysis of the hedging strategies. Recall that the claim sizes are Gamma(α, β) distributed.

CVA, where the amount CVA_0 is invested in the bank account and where one does not invest in the CDS at all ($V_0(\xi) = \mathrm{CVA}_0$ and $\xi^1_t \equiv 0$). In order to measure the performance of a hedging strategy we consider the value of the hedged CVA position, which is given by

$$e_t := \mathrm{CVA}_t - \Big(\mathrm{CVA}_0 + \int_0^t \xi^1_s \mathrm{d}S_s \Big), \quad 0 \le t \le T. \tag{29}$$

In the sequel we refer to the process $(e_t)_{0 \le t \le T}$ in (29) as the *tracking error*. Note that a positive value of e_T corresponds to a loss for the insurance company. In our experiments we assume that the hedging portfolio is re-balanced approximately every two weeks. More frequent re-balancing is not practically feasible for insurance companies as the total claim amount is hard to evaluate.

In Figure 2 we use the parameter set corresponding to Case 1. The plot displays 2000 trajectories of the tracking error, first for $\xi^1 = 0$ (unhedged CVA), second for the static CDS strategy $\xi^1 = \mathrm{CVA}_0 / \zeta$ and third for the dynamic $\mathbf{Q}$-mean-variance minimizing strategy $\xi^1 = \xi^{1,*}$ from Theorem 4.3.

From Figure 2 it is evident that for all three strategies the tracking error jumps at τ_R, but the form of the jumps is very different. In the unhedged-CVA case the jump is always upwards and the size of the jump is equal to the replacement cost for the reinsurance contract. In this case a default of R is relatively expensive: the maximum loss that the insurance company incurs is around EUR 40, which is roughly three times the initial value of the reinsurance contract (A numerical computation in this example gave $V_0^\phi \approx 11.89$). In the middle panel we give the tracking error for the static CDS hedging strategy. We observe either a loss (under-hedging) or a profit (over-hedging). The maximum loss (and profit) is around EUR 20 which implies that static hedging is an improvement over the unhedged CVA , but the tracking error still shows a high variability. The dynamic mean-variance minimizing strategy on the other hand significantly reduces the variability of the tracking error as it is clearly displayed in the lower panel. We conclude that this strategy out-performs the other hedging approaches by a large margin. The difference in the performance of the hedging strategies is illustrated further in Figure 3 where we plot the density of the tracking error e_T conditional on $\{\tau_R < T\}$. For a good hedging strategy the density of the tracking error should be concentrated around zero with a small mass in the tails. This is the case for the mean-variance minimizing strategy. The densities for the two other strategies have much larger mass in the tails. The shape of these densities is identical, but that corresponding to the static CDS strategy is shifted to the left, which results in a lower value of $\mathbb{E}^{\mathbf{Q}}\left[e_T^2\right]$. The value of the L^2-norm of e_T for all three strategies is given in Table 2.

In order to explain the superior performance of the dynamic strategy we plot in Figure 4 two trajectories $\xi^{1,*}_\cdot(\omega)$ of the optimal strategy. The solid line corresponds

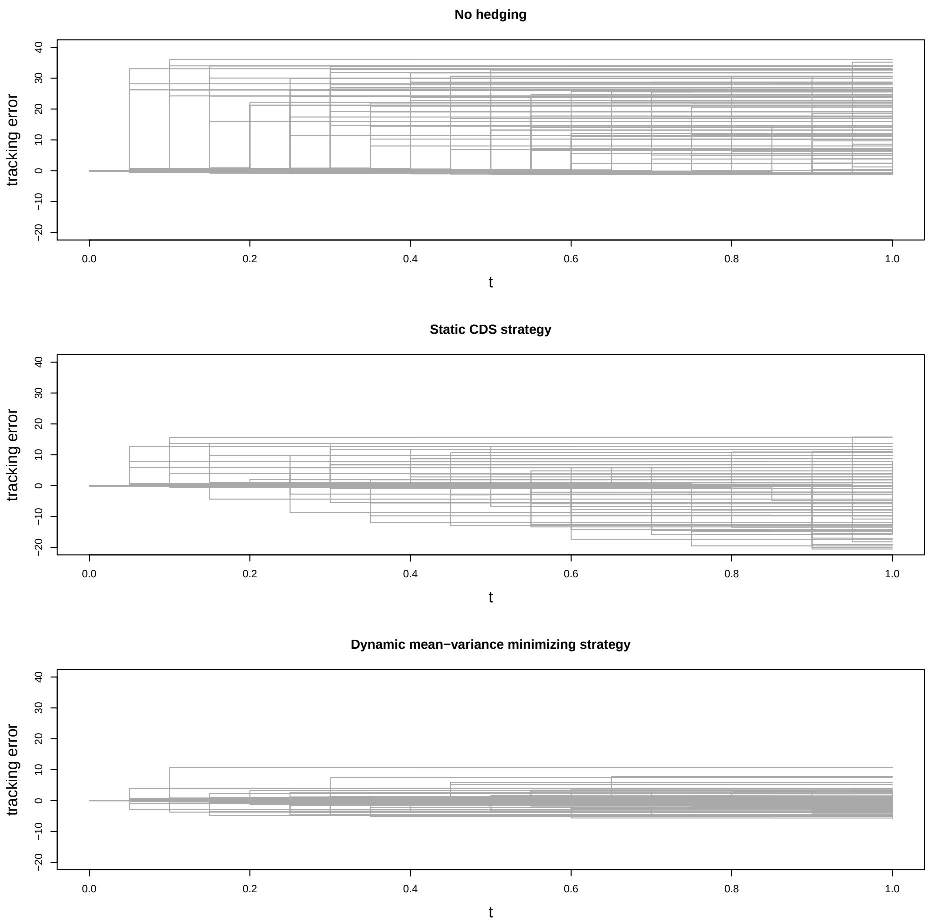

Fig. 2. *Performance of various hedging strategies for the parameters in Case 1: the upper panel corresponds to no hedging, the middle panel to static hedging and the lower panel to dynamic mean-variance hedging.*

Strategy	$\mathbb{E}^{\mathbf{Q}}\left[e_T^2\right]$
No hedging	22.65
Static CDS hedging	4.54
Dynamic mean-variance minimizing	0.62

Table 2
L^2-norm of the tracking error e_T in Case 1.

to a trajectory of the claim amount process with a large loss, the dashed line to a trajectory with small loss. We compare these strategies to the static hedging strategy which is constant over time (grey line). We see that the optimal hedge ratio is quite sensitive with respect to the evolution of the underlying loss process.

In Case 2 we consider the situation where claims arrive less frequently but have on average a higher size. In this case hedging is more difficult, but the mean-variance minimizing strategy still outperforms the other approaches, as is clearly seen from

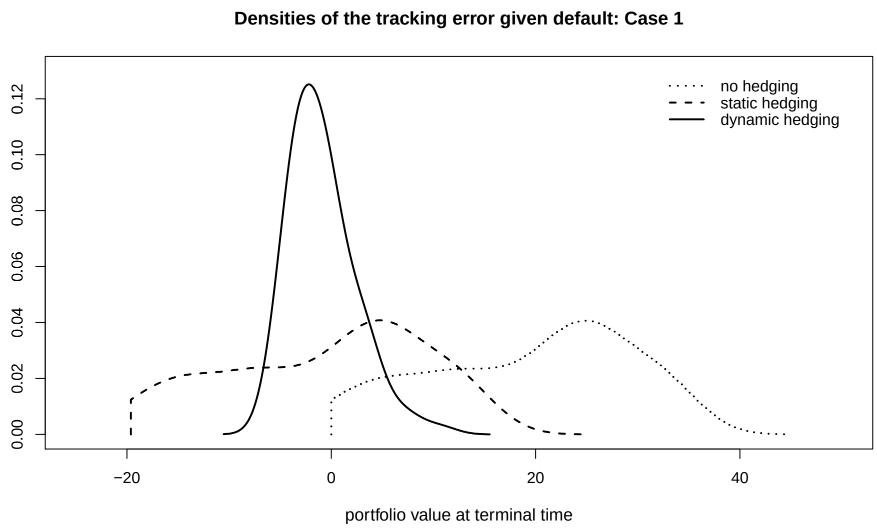

FIG. 3. *Densities the tracking error e_T given default in Case 1.*

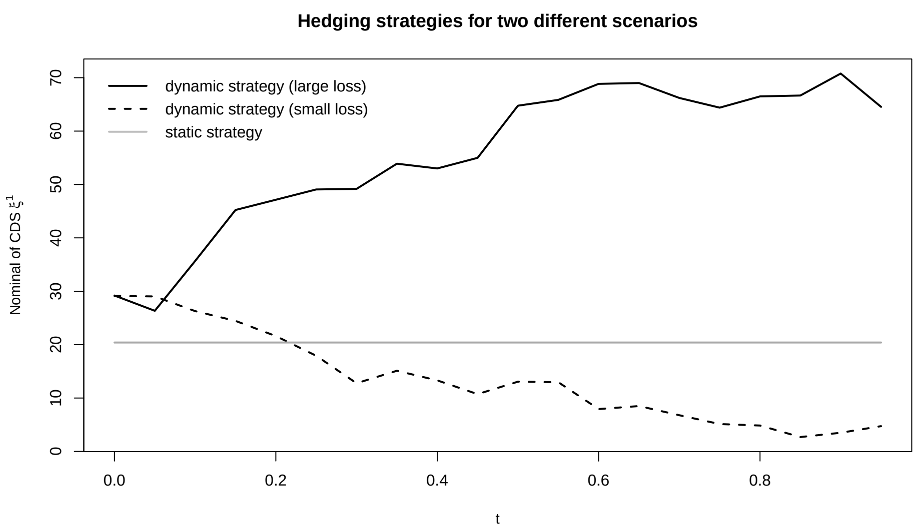

FIG. 4. *Optimal strategies for two scenarios with a large loss and a low loss respectively and the constant strategy for the parameter in Case 1.*

Figure 5. Moreover, for the mean-variance minimizing strategy the L^2-norm of the tracking error is considerably smaller than for the other strategies, see Table 3 for details. In Case 3 we consider the situation where the loss and the default intensities are correlated but there is no pricing contagion ($\gamma = 0$), that is the loss intensity does not jump at time τ_R. Here the wrong way risk arises from correlation only. As the effect of price contagion dominates the correlation effect, the L^2-norm of the tracking

error for all strategies is considerably smaller than in Case 1 and Case 2. However, Figure 6 and Table 4 confirm the relative performance of the strategies for this case as well. In the general version of the model with price contagion and correlation the qualitative results on the behaviour of the tracking error are similar to the ones described so far; we omit the details.

Summarizing, our results show that dynamic CDS trading strategies have the potential to significantly reduce reinsurance counterparty risk, both compared to a static hedging strategy and to the case where the insurance company does not hedge at all.

Strategy	$\mathbb{E}^{\mathbf{Q}}\left[e_T^2\right]$
No hedging	39.78
Static CDS hedging	17.82
Dynamic mean-variance minimizing	2.17

TABLE 3
L^2-norm of the tracking error in Case 2.

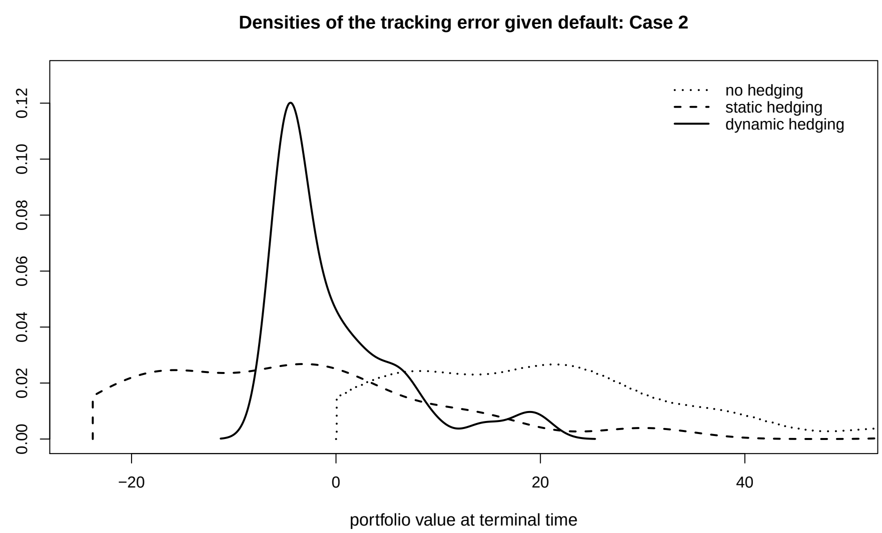

FIG. 5. *Densities the tracking error at terminal time given default in Case 2.*

	$\mathbb{E}^{\mathbf{Q}}\left[e_T^2\right]$
No hedging	12.75
Static CDS hedging	4.57
Dynamic mean-variance minimizing	0.97

TABLE 4
L^2-norm of the tracking error in Case 3.

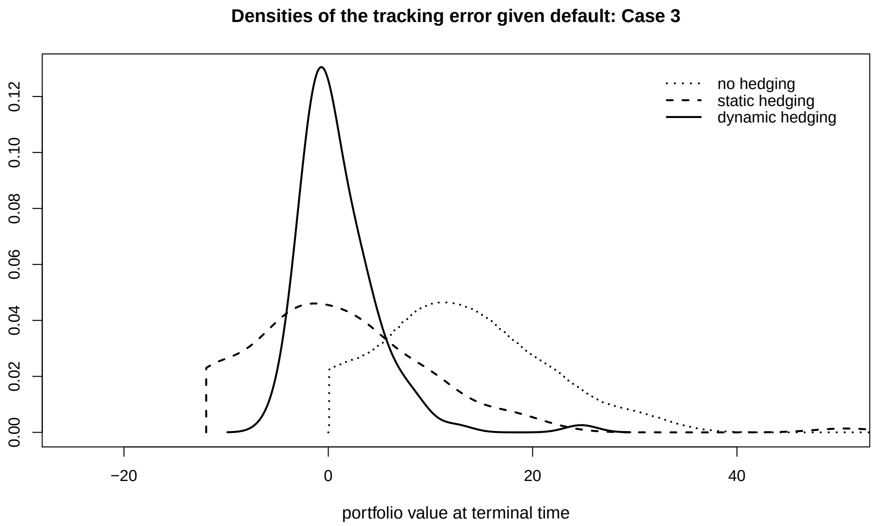

Fig. 6. *Densities the tracking error at terminal time given default in Case 3.*

Acknowledgements. The authors are grateful for useful comments from Hansjörg Albrecher. Support by the Vienna Science and Technology Fund (WWTF) through project MA14-031 is gratefully acknowledged. A part of this article was written while K. Colaneri was affiliated with the School of Mathematics, University of Leeds, LS2 9JT, Leeds, UK. The work of C. Ceci and K. Colaneri was partially supported by INdAM-GNAMPA through projects UFMBAZ- 2017/0000327 and UFMBAZ-2019/000436.

Appendix A. The martingales M^{CL} and S. In the sequel we provide detailed computations for the dynamics of the martingale M^{CL}. We start with the martingale M^{CL}. For every $0 \le t \le T$ we have that

$$M_t^{\mathrm{CL}} = \int_0^t e^{-rs} v^{\phi}(s, \widetilde{L}_{s^-}, \widetilde{X}_{s^-} + \gamma^X(\widetilde{X}_{s^-}))\mathrm{d}H_s^R + e^{-rt}\delta^R(1 - H_t^R) f^{\mathrm{CVA}}(t, \widetilde{L}_t, \widetilde{X}_t, Y_t),$$

so that

$$\begin{aligned}\mathrm{d}M_t^{\mathrm{CL}} &= e^{-rt}(v^{\phi}(t, \widetilde{L}_{t^-}, \widetilde{X}_{t^-} + \gamma^X(\widetilde{X}_{t^-}) - f^{\mathrm{CVA}}(t, \widetilde{L}_{t^-}, \widetilde{X}_{t^-}, Y_t))\mathrm{d}H_t^R \\ &\quad - re^{-rt}(1 - H_{t-}^R) f^{CVA}(t, \widetilde{L}_{t-}, \widetilde{X}_t, Y_t)\mathrm{d}t + e^{-rt}(1 - H_{t-}^R)\mathrm{d}f^{\mathrm{CVA}}(t, \widetilde{L}_t, \widetilde{X}_t, Y_t)\,.\end{aligned}$$

Recall that by Proposition 3.3, f^{CVA} is a smooth solutions of the PIDE (11), therefore it has the necessary regularity to apply the Itô formula. This gives

$$
\begin{aligned}
&\mathrm{d}f^{\mathrm{CVA}}(t,\widetilde{L}_t,\widetilde{X}_t,Y_t) = \\
&\left(\frac{\partial f^{\mathrm{CVA}}}{\partial x}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t)\sigma^X(\widetilde{X}_t) + \frac{\partial f^{\mathrm{CVA}}}{\partial y}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t)\sigma^Y(Y_t)\rho\right)\mathrm{d}W^1_t \\
&+ \frac{\partial f^{\mathrm{CVA}}}{\partial y}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t)\sigma^Y(Y_t)\sqrt{1-\rho^2}\,\mathrm{d}W^2_t \\
&+ \int_{\mathbb{R}^+}\left(f^{\mathrm{CVA}}(t,\widetilde{L}_{t-}+z,\widetilde{X}_t,Y_t) - f^{\mathrm{CVA}}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t)\right)m^L(\mathrm{d}s,\mathrm{d}z) \\
&+ \left(\frac{\partial f^{\mathrm{CVA}}}{\partial t}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t) + b^X(\widetilde{X}_t)\frac{\partial f^{\mathrm{CVA}}}{\partial x}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t)\right. \\
&\quad + b^Y(Y_t)\frac{\partial f^{\mathrm{CVA}}}{\partial y}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t) + \frac{1}{2}(\sigma^X(\widetilde{X}_t))^2\frac{\partial^2 f^{\mathrm{CVA}}}{\partial x^2}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t) \\
&\quad \left. + \frac{1}{2}(\sigma^Y(Y_t))^2\frac{\partial^2 f^{\mathrm{CVA}}}{\partial y^2}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t) + \rho\sigma^X(\widetilde{X}_t)\sigma^Y(Y_t)\frac{\partial^2 f^{\mathrm{CVA}}}{\partial x\partial y}(t,\widetilde{L}_{t-},\widetilde{X}_t,Y_t)\right)\mathrm{d}t.
\end{aligned}
$$

Now using the fact that f^{CVA} solves equation (11) we get that M^{CL} satisfies equation (6). Similar computations can be performed for the martingale S, we omit the details.

REFERENCES

[1] H. Albrecher, J. Beirlant, and J. Teugels, *Reinsurance: Actuarial and Statistical Aspects*, Wiley, 2017.

[2] C. Bernard and M. Ludkovski, *Impact of counterparty risk on the reinsurance market*, North American Actuarial Journal, 16 (2012), pp. 87–111.

[3] F. Biagini, C. Botero, and I. Schreiber, *Risk-minimization for life insurance liabilities with dependent mortality risk*, Mathematical Finance, 27 (2017), pp. 505–533.

[4] T. Bielecki and M. Rutkowski, *Credit Risk: Modeling, Valuation and Hedging*, Springer Science & Business Media, 2004.

[5] L. Bo and C. Ceci, *Locally risk-minimizing hedging of counterparty risk for portfolio of credit derivatives*, Applied Mathematics & Optimization, (2019), pp. 1–52.

[6] N. Bodoff, *Reinsurance credit risk: A market-consistent paradigm for quantifying the cost of risk*, Variance Advancing the Science of Risk, Casualty Actuarial Society, 7 (2013), pp. 11–28.

[7] P. Brémaud, *Point Processes and Queues: Martingale Dynamics*, Springer, New York, 1981.

[8] D. Brigo and F. Mercurio, *Interest rate models-theory and practice: with smile, inflation and credit*, Springer Science & Business Media, 2007.

[9] D. Brigo, M. Morini, and A. Pallavicini, *Counterparty credit risk, collateral and funding: with pricing cases for all asset classes*, vol. 478, John Wiley & Sons, 2013.

[10] J. Cai, C. Lemieux, and F. Liu, *Optimal reinsurance with regulatory initial capital and default risk*, Insurance: Mathematics and Economics, 57 (2014), pp. 13–24.

[11] C. Ceci, K. Colaneri, and A. Cretarola, *Hedging of unit-linked life insurance contracts with unobservable mortality hazard rate via local risk-minimization*, Insurance: Mathematics and Economics, 60 (2015), pp. 47–60.

[12] C. Ceci, K. Colaneri, and A. Cretarola, *Unit-linked life insurance policies: optimal hedging in partially observable market models*, Insurance: Mathematics and Economics, 76 (2017), pp. 149–163.

[13] CEIOPS, *CEIOPS' advice for Level 2 implementing measures on Solvency II: SCR Standard Formula, Article 109(1c), Correlations-Counterparty default risk module*, tech. report, Committee of European Insurance and Occupational Pensions Supervisors, October 2009. CEIOPS-DOC-23/09.

[14] K. Colaneri and R. Frey, *Classical solutions of the backward PIDE for a Markov point process with characteristics modulated by a jump diffusion*, Preprint ArXiv, https://arxiv.org/pdf/1903.07492.pdf, (2019).

[15] S. Crépey, *Bilateral counterparty risk under funding constraints—Part I: Pricing*, Mathematical Finance, 25 (2015), pp. 1–22.
[16] S. Crépey, *Bilateral counterparty risk under funding constraints—Part II: CVA*, Mathematical Finance, 25 (2015), pp. 23–50.
[17] M. Dahl and T. Møller, *Valuation and hedging of life insurance liabilities with systematic mortality risk*, Insurance: Mathematics and Economics, 39 (2006), pp. 193–217.
[18] D. Duffie, J. Pan, and K. Singleton, *Transform analysis and asset pricing for affine jump-diffusions*, Econometrica, 68 (2000), pp. 1343–1376.
[19] M. Flower, M. Afify, I. Cook, V. Gosrani, G. James, P. Koulovasilopoulos, J. Lincoln, D. Maneval, and J. Robinson, *Reinsurance counterparty credit risks: Practical Suggestions For Pricing, Reserving and Capital Modelling.* GIRO working paper, Institute and Faculty of Actuaries, 2007.
[20] R. Frey and L. Rösler, *Contagion Effects and Collateralized Credit Value Adjustments for Credit Default Swaps*, International Journal of Theoretical and Applied Finance, 17 (2014), p. 1450044.
[21] P. Glasserman, *Monte Carlo methods in financial engineering*, Springer, 2003.
[22] J. Grandell, *Aspects of risk theory*, Springer Science & Business Media, 2012.
[23] J. Gregory, *Counterparty credit risk and credit value adjustment: A continuing challenge for global financial markets*, John Wiley & Sons, 2012.
[24] Y. Kravych and P. Shevchenko, *Managing exposure to reinsurance credit risk*, in ASTIN Colloquium 2011, Madrid, 2011, pp. 1–17. http://www.actuaries.org/ASTIN/Colloquia/Madrid/Papers/Krvavych_Shevchenko.pdf.
[25] A. McNeil, R. Frey, and P. Embrechts, *Quantitative Risk Management: Concepts, Techniques and Tools - revised edition*, Princeton University Press, 2015.
[26] T. Møller, *Risk-minimizing hedging strategies for insurance payment processes*, Finance and Stochastics, 5 (2001), pp. 419–446.
[27] B. Oksendal, *Stochastic differential equations: an introduction with applications*, Springer Science & Business Media, 2013.
[28] M. Schweizer, *A guided tour through quadratic hedging approaches*, in Option Pricing, Interest Rates and Risk Management, E. Jouini, J. Cvitanic, and M. Musiela, eds., Cambridge University Press, 2001, pp. 538–574.
[29] R. Shaw, *The modelling of reinsurance credit risk.* GIRO working paper, Institute and Faculty of Actuaries, 2007.
[30] N. Vandaele and M. Vanmaele, *A locally risk-minimizing hedging strategy for unit-linked life insurance contracts in a Lévy process financial market*, Insurance: Mathematics and Economics, 42 (2008), pp. 1128–1137.