Market Illiquidity as a Source of Model Risk in Dynamic Hedging

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Abstract

In the present paper we study market illiquidity as a particular source of model risk in the hedging of derivatives. We depart from the usual Black-Scholes framework, where it is assumed that option hedgers are small traders, and consider a model where the implementation of a hedging strategy affects the price of the underlying security. We derive a formula for the feedback-effect of dynamic hedging on market volatility and present a formula for the hedging error due to market illiquidity. We go on and characterize perfect hedging strategies by a nonlinear version of the Black-Scholes PDE. We relate this PDE to other models for the risk-management of derivatives under market frictions and present some simulations.

Key words: Option hedging, illiquid markets, large trader models, feedback-effects, nonlinear Black-Scholes equation

1 Introduction

The recent turbulences on financial markets and in particular the events surrounding the LTCM-debacle have made market liquidity an issue of high concern to investors and risk managers. The latter group in particular realized that financial models which are based on the assumption that an investor can trade large amounts of an asset without affecting its price (perfectly liquid markets) may fail miserably in circumstances where market liquidity vanishes. If we consider model risk to be the risk that a financial institution incurs a loss because some of the key assumptions underlying its risk-management models are not met in practice, losses due to vanishing market liquidity form a prime example of model risk. Understanding the robustness of models used for hedging and risk-management purposes with respect to the assumption of perfectly liquid markets is therefore an important issue in the analysis of model risk in general.

The present paper studies the hedging of derivatives via dynamic trading strategies in markets which are not perfectly liquid. More precisely, we consider a model where the implementation of a hedging strategy affects the price process of the underlying asset. We derive a formula for the feedback-effect of dynamic hedging on market volatility. In our framework, market volatility is increased if the hedger uses a strategy which requires additional buying if the price rises such as the standard hedging strategy for a European call; it is decreased if the hedger has to sell in reaction to rising prices of the underlying. In

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this framework a standard Black-Scholes delta-hedging strategy does no longer eliminate all risk of a derivative position. We derive an explicit formula for the loss of an investor who relies on a standard Black-Scholes delta-hedge to cover his position. This formula yields some interesting insights into the determinants of model risk related to the lack of market liquidity.

We go on and study, if our trader can replicate the payoff of a derivative, if he uses a hedging strategy different from the standard Black-Scholes strategy. We develop a characterization of perfect hedging strategies by a nonlinear version of the standard Black Scholes partial differential equation (PDE). We relate this PDE to similar equations obtained by other authors in the analysis of option hedging in the presence of market incompleteness and/or transaction cost. Finally, we present some simulations which illustrate quantitatively the hedge-cost in our framework.

The hedging of derivatives in markets which are not perfectly liquid has been the focus of a number of recent studies. Here we mention only the interesting contributions by Jarrow (1994), Frey and Stremme (1997), Platen and Schweizer (1998), Frey (1998), Sircar and Papanicolaou (1998), Schönbucher and Willmott (1998) and Bank (1999). Our paper builds on this work and relates it to the theme of model risk.

2 The model

We are working in a stylized market with two traded assets: a riskless one (typically a bond or a money market account), called the bond and a risky one (typically a stock or stock index) referred to as the stock. We take the bond as numeraire. Moreover, we assume that the market for the bond is perfectly elastic, i.e. investors can buy or sell arbitrarily large quantities of this security without affecting its price. This reflects the fact that money markets are usually far more liquid than the market for the stock we have in mind in this study. As usual the price of the stock, accounted in units of the numeraire, is modelled as a stochastic process $(S_t)_t$ on some underlying filtered probability space $(\Omega, \mathcal{F}, P)$, $(F_t)_t$.

We consider an agent who wants to replicate a derivative contract on the stock with maturity date T using a dynamic trading strategy in stock and bond. His strategy is given by a pair $(\alpha_t, \beta_t)_t$ of adapted processes, with α_t and β_t denoting the number of shares respectively the number of bonds in the portfolio at time t. In contrast to standard derivative asset analysis we assume that our hedger is a *large trader*, i.e. the implementation of his hedging strategy affects the price of the stock. More precisely, we assume that the stock price rises (falls) if he buys (sells) additional shares of the stock. This assumption is in line with economic intuition on the price impact of large trades; it is also supported by empirical evidence on price impact of large block transactions as given for instance in Holthausen and Leftwich (1987).

In principle there are two different avenues we could follow in modelling the stock-price dynamics in our setting. One the one hand we could develop a full economic equilibrium model with different types of traders. This requires specifying the information of these traders, their motives for trading and in particular their way of learning and expectation-formation. In such a framework the impact of trading on prices, and hence market liquidity can be endogenously derived. This type of modelling is certainly called for if we want to understand the economic determinants of market liquidity. Interesting examples of this line of research are Kyle (1985) or Back (1992). These papers analyze how the presence of agents with informational advantages over a price-setting market maker (insider information) affects market liquidity. In their framework markets are not perfectly liquid, as the market maker changes his perception of the asset value and hence his quotes in reaction to the bids and offers he observes.

In the present paper we take a simpler route and model directly the asset price dynamics which result if the large trader chooses a given stock-trading strategy α. In particular, form and size of the price-impact of our hedger's trades are not derived but exogenously imposed.[1] Obviously this simplifies the analysis considerably. As we are mainly concerned with model-risk and the robustness of dynamic hedging with respect to the assumption of perfectly liquid markets we think that this simpler approach is appropriate. Moreover, the primitives of our analysis are at least in principle observable which facilitates the application of our results.

Before we can specify the stock-price dynamics we need to impose some technical restrictions on the class of stock trading strategies permissible for our trader. Throughout the paper we assume that

A1) The stockholdings $(\alpha_t)_t$ are left-continuous (i.e. $\alpha_t = \lim_{s \overset{<}{\to} t} \alpha_s$).

A2) The right-continuous process α^+ with $\alpha_t^+ = \lim_{s \overset{>}{\to} t} \alpha_s$ is a semimartingale.[2]

A3) The downward-jumps of our strategy are bounded: $\Delta\alpha_t^+ := \alpha_t^+ - \alpha_t > -1/\overline{\rho}$ for some $\overline{\rho} > 0$.

Most of the time we will work with trading strategies which are smooth functions of the stock-price as in the standard Black-Scholes model. For these strategies the above assumptions are always satisfied.

Our model can be viewed as a pertubation of the standard Black-Scholes model. The size of this pertubation is controlled by a parameter ρ (the market liquidity parameter). In fact, if the hedger does not trade, i.e. if $\alpha_t \equiv 0$, or if $\rho = 0$ the asset price simply follows a Black-Scholes model with some reference volatility σ. In the sequel we denote the asset price process which results if the liquidity parameter takes a certain value ρ and if the large trader uses a particular trading strategy α by $S_t(\rho, \alpha)$. In the following Assumption we describe the dynamics of $S_t(\rho, \alpha)$ by a stochastic differential equation (SDE).

A4) Consider some Brownian motion W on our underlying probability space and two constants σ and ρ with $\sigma > 0$ and $0 \leq \rho < \overline{\rho}$. Suppose that the large trader uses a stock-trading strategy $(\alpha)_t$ satisfying Assumptions A1, A2 and A3. Then the asset price process solves the following stochastic differential equation (SDE)

$$dS_t = \sigma S_{t-} dW_t + \rho S_{t-} d\alpha_t^+ , \tag{2.1}$$

where S_{t-} denotes the left limit $\lim_{s \overset{<}{\to} t} S_t$.

Note that $1/(\rho S_t)$ measures the *depth of the market* at time t, i.e. the size of the change in the large trader's stock position which causes the price to move by one unit of account. The following example is helpful in understanding the implications of the stock-price dynamics (2.1). Assume that the large trader holds K shares of the stock

[1] It is however possible to derive the asset price dynamics we use in our analysis from stylized microeconomic models; see Föllmer and Schweizer (1993) or Frey and Stremme (1997).

[2] An introduction into semimartingale theory can be found in Protter (1992).

$(0 < K < 1/\overline{\rho})$, and that he protects his position by a limit order with limit $\overline{S}$, i.e. he gives an order to sell the stocks as soon as the stock trades below $\overline{S}$. In a perfectly liquid market with continuous trading this guarantees that the value of his portfolio can never drop below $K\overline{S}$. Formally we can describe his trading strategy as follows: Define the stopping time τ by $\tau = \inf\{t > 0, S_t < \overline{S}\}$ and the trading strategy α by $\alpha_t = K$, $0 \le t \le \tau$ and $\alpha_t = 0$ for $t > \tau$. Then α is left-continuous, and the right continuous version α^+ has a jump of size $(-K)$ at time τ. Under Assumption A4) this causes a drop in the share price at time τ; more precisely we get from (2.1) that $S_\tau = S_{\tau-} - \rho S_{\tau-} K < S_{\tau-}$. Note that in our framework the value of the portfolio is not perfectly protected by the limit order: we have $S_{\tau-} = \overline{S}$ and the trader receives only

$$KS_\tau = K\overline{S} - \rho\overline{S}K^2 < K\overline{S}$$

in return for his shares. Seasoned traders know of course, that limit orders provide only partial protection in an illiquid market. The ability to reproduce this effect is an interesting feature of our model.

Obvious extensions of our model are the introduction of a liquidity coefficient ρ which depends on the current stock price or which is even stochastic. If we allow for these slight generalizations the dynamics (2.1) subsume most of the models that have been proposed in the recent literature for the analysis of option hedging in illiquid markets.

To make the model operational one has to determine ρ. Here various approaches come to mind. To begin with, we think that a seasoned trader has a good feeling for the price-impact of his trades which might be turned into estimates of ρ. On the academic side there have been several studies on the price-impact of large block transactions, which give a range for the values of the market liquidity parameters. Here we mention only the contribution of Holthausen and Leftwich (1987); further references can be found in Almgren and Chriss (1998). Finally, one of the main applications of our model is to understand the robustness of hedging strategies for a given derivative portfolio with respect to the lack of market liquidity. For this purpose it is enough to have a crude range for possible values of ρ.

3 Market illiquidities as a source of model risk in dynamic hedging

3.1 Dynamic hedging: Basic concepts revisited

In the sequel we discuss some modifications to basic notions in derivative asset analysis necessary to account for the fact that our hedger is a large investor. Consider a trading strategy $\xi = (\alpha, \beta)$ satisfying Assumptions A1, A2 and A3.

VALUE PROCESS: In defining the value of the large trader's position we have to distinguish between the *paper value* or *mark to market value* and the *liquidation value* of his position. The mark to market value of his portfolio at time t is simply given by $V_t^M := \alpha_t S_t(\rho, \alpha) + \beta_t$, i.e. by valuing the position with the current market prices. The liquidation value of a portfolio corresponds to the funds an investor obtains when actually selling his stockholdings; as shown by the analysis of limit orders in Section 2, if $\rho > 0$ the liquidation value is lower than the paper value and the large trader incurs a liquidation cost. The exact liquidation value of a portfolio is difficult to determine as it depends on the liquidation strategy chosen by the large trader. In our present discussion of dynamic hedging with market illiquidity we restrict ourselves to mark-to-market values as this allows for the derivation of very clearcut results. Schönbucher and Willmott (1998) discusses some of the conceptual difficulties which arise if we try to incorporate liquidation costs into our analysis of dynamic hedging; "optimal" liquidation strategies are for instance discussed in Almgren and Chriss (1998).

Gains from trade and selffinancing strategies: As in standard derivative asset analysis the gains from trade from a stock-trading strategy α are given by $G_t := \int_0^T \alpha_s dS_s(\rho,\alpha)$; note however, that in our situation the stock price process S depends on the chosen strategy α. We call a strategy selffinancing if $V_t^M = V_0^M + G_t$ for all $0 \le t \le T$. As usual a stock-trading-strategy α satisfying Assumptions A1, A2 and A3 and an initial investment V_0 define a unique selffinancing strategy (α,β) in stock and bond. Hence when restricting ourselves to selffinancing strategies we do not have to specify the amount of bonds in the portfolio.

Tracking error: We now introduce the tracking error of our strategies. This number measures the the difference between the value of a selffinancing hedging strategy designed to replicate a derivative and the payoff of this derivative at maturity; it is therefore an essential quantity if we want to assess the model risk that stems from applying the standard Black-Scholes theory in markets which are not perfectly liquid. Consider some derivative security with payoff $h(S_T)$ and a selffinancing trading strategy with initial value V_0 and stock-trading-strategy α. The tracking error e_T^M of this strategy is defined by

$$e_T^M := h(S_T(\rho,\alpha)) - V_T^M = h(S_T(\rho,\alpha)) - \left(V_0 + \int_0^T \alpha_s dS_s(\rho,\alpha)\right). \tag{3.1}$$

A positive (negative) value of e_T^M obviously indicates that we made a loss (profit) on our hedge. The superscript V^M and e^M has been introduced to remind the reader that both quantities are defined in terms of mark-to-market values and do not account for liquidation cost.

3.2 Dynamic hedging and volatility

In this subsection we determine the dynamics of the asset price and in particular its volatility assuming that the large trader's stock-trading strategy is given by a smooth function ϕ of time and the current stock price. This is a prerequisite for studying any kind of option replication in our setting. We make the following regularity assumptions on ϕ.

A5) The function $\phi:\ [0,T]\times\mathbb{R}^+ \to \mathbb{R}$ is of class $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^+)$. Moreover, $\rho S\phi_{SS}(t,S) < 1$ for all $(t,S) \in [0,T]\times\mathbb{R}^+$.

We have

Proposition 3.1. *Suppose that the large trader uses a stock-trading-strategy of the form $\alpha_t = \phi(t,S_t)$ for a function ϕ satisfying Assumption A5 and that the stock price process $S_t = S_t(\rho,\alpha)$ follows an Itô process of the form*

$$dS_t = v(t,S_t)S_t dW_t + b(t,S_t)dt \tag{3.2}$$

for two functions v and b. Then we have under Assumption A4

$$v(t,S) = \frac{\sigma}{1-\rho S\phi_S(t,S)} \quad \textit{and} \tag{3.3}$$

$$b(t,S) = \frac{\rho S}{1-\rho S\phi_S(t,S)}\left(\phi_t(t,S) + \frac{\sigma^2 S^2}{(1-\rho S\phi_S(t,S))^2}\right). \tag{3.4}$$

The proof can be found in Appendix A.1. Proposition 3.1 describes the feedback-effect from dynamic hedging on volatility: by the trading-activity of the large investor the constant volatility σ is transformed into the time- and space dependent volatility $v(t,S)$. Obviously, $v(t,S) > \sigma$ if $\phi_S(t,S) > 0$, i.e. if the trader uses a positive feedback strategy which calls for additional buying if the stock price rises. This property is typical for hedging strategies for derivatives with a convex terminal payoff such as European call or put options; see for instance El Karoui, Jeanblanc-Picqué, and Shreve (1998). On the other hand, if our trader uses a contrarian strategy, i.e. if $\phi_S(t,S) < 0$ we have $v(t,S) < \sigma$ and the volatility is decreased.

In our model the widespread use of positive feedback strategies is very likely to have a destabilizing effect on asset prices. This is in line with anecdotal evidence from markets. In fact, the severity of the stock-market crash from October 1987 is often attributed to the fact that a large part of the market followed so-called portfolio insurance strategies.[3] For a detailed discussion of the relation between dynamic portfolio insurance and market volatility we refer the reader to Brennan and Schwartz (1989), Gennotte and Leland (1990) and Frey and Stremme (1997).

3.3 Model-risk in illiquid markets: the case of Black-Scholes strategies

We now derive an explicit expression for the profit or loss (P&L) of a trader, who uses a standard Black-Scholes strategy to hedge a derivative with payoff $h(S_T)$ in our illiquid-market-model. This P&L is measured by the tracking error e_T^M introduced in (3.1). Obviously, an explicit formula for e_T^M is of great help in analyzing the robustness of the standard Black-Scholes hedging with respect to the lack of market liquidity.

Denote by u^{BS} the Black-Scholes price of our derivative corresponding to the reference volatility σ. It is well-known that u^{BS} can be characterized by the following PDE (recall that the interest-rate is assumed to be zero)

$$u_t^{\mathrm{BS}}(t,S) + \frac{1}{2}\sigma^2 S^2 u_{SS}^{\mathrm{BS}}(t,S) = 0\,, \quad u(T,S) = h(S)\,. \tag{3.5}$$

As usual the subscripts denote partial derivatives. A derivation can be found in any standard textbook in Finance such as Duffie (1992) or Musiela and Rutkowski (1997). The standard Black-Scholes delta-hedge corresponding to the volatility σ now consists of running a selffinancing hedging strategy with

$$\text{initial value } V_0 = u^{\mathrm{BS}}(0,S_0) \text{ and stock-position } \alpha_t^{\mathrm{BS}} = u_S^{\mathrm{BS}}(t,S_t)\,. \tag{3.6}$$

Proposition 3.2. *Suppose that the large trader uses the Black-Scholes trading strategy (3.6) and that the function $u_S^{\mathrm{BS}}(t,S)$ satisfies A5). Then the tracking error e_T^M is given by*

$$e_T^M = \int_0^T \frac{1}{2}\sigma^2\left(\frac{1}{(1-\rho S_s u_{SS}^{\mathrm{BS}}(s,S_s))^2} - 1\right) S_s^2 u_{SS}^{\mathrm{BS}}(s,S_s)\,ds\,, \tag{3.7}$$

[3] Portfolio Insurance strategies are dynamic hedging strategies of the positive feedback-type which are closely related to replicating strategies for European put options; see for instance Chapter 14 of Hull (1997) for details.

where S_t is short for the stock price $S_t(\rho, \alpha^{\mathrm{BS}})$.

For u^{BS} to satisfy Assumption A5 it suffices that the hedged payoff is sufficiently smooth. As shown in Frey and Stremme (1997), this is for instance the case if our hedger holds a well-diversified portfolio of derivatives containing a multitude of different contracts.

We give the simple proof of Proposition 3.2 in Appendix A.2, as it provides some insights into the nature of model risk due to volatility misspecification in general. The idea behind Proposition 3.2 goes back El Karoui and Jeanblanc-Piqué (1990); interesting generalizations and applications to the analysis of model risk and hedging under stochastic volatility are given in Avellaneda, Levy, and Paras (1995), Lyons (1995), Frey and Stremme (1997), El Karoui, Jeanblanc-Picqué, and Shreve (1998), or Bossy et.al. (1999), among others.

Note that the tracking-error e_T^M is always positive such that our Black-Scholes hedging will result in a loss: if $u_{SS}^{\mathrm{BS}}(t,S) > 0$ we have $1/(1-\rho S_s u_{SS}^{\mathrm{BS}}(t,S))^2 > 1$ and if $u_{SS}^{\mathrm{BS}}(t,S) < 0$ we have $1/(1-\rho S_s u_{SS}^{\mathrm{BS}}(t,S))^2 < 1$, such that the integrand in (3.7) is always nonnegative. Hence in markets which are not perfectly liquid Black-Scholes hedging is costly.[4]

By (3.7) the tracking error takes the form of a cumulative dividend stream with instantaneous dividend equal to

$$\delta(t, S_t; \rho) := \frac{1}{2}\sigma^2 \left(\frac{1}{(1-\rho S_t u_{SS}^{\mathrm{BS}}(t,S_t))^2} - 1 \right) S_t^2 u_{SS}^{\mathrm{BS}}(t,S_t)\,,$$

i.e. the cost due to market illiquidity the hedger incurs in the period $[t, t+\varepsilon]$ is approximately equal to $\varepsilon\delta(t, S_t; \rho)$. It is instructive to consider the form of this dividend stream for small values of ρ, i.e. for relatively liquid markets. As $\delta(t, S; 0) = 0$ we get

$$\delta(t, S, \rho) \approx \rho \frac{\partial}{\partial \rho}_{\big|\rho=0} \delta(t, S, \rho) = \rho S \left(\sigma S u_{SS}^{\mathrm{BS}}(t,S)\right)^2 .$$

It can be shown that for ρ small we have

$$e_T^M \approx \int_0^T \rho S_t^{\mathrm{BS}} \left(\sigma S_t^{\mathrm{BS}} u_{SS}^{\mathrm{BS}}(t, S_t^{\mathrm{BS}})\right)^2 dt\,, \tag{3.8}$$

where $(S_t^{\mathrm{BS}})_{0\le t\le T}$ now represents a path of the standard Black-Scholes model with $\rho = 0$. Obviously, relation (3.8) can be used for simulating the tracking error.

By Itô's formula $\left(\sigma S_t u_S^{\mathrm{BS}} S(t,S_t)\right)^2$ is the instantaneous quadratic variation of the large trader's stock position and therefore a measure for his local trading activity; the expression ρS_t is the inverse of the market depth at time t. Hence the loss the hedger incurs is in first order proportional to his local trading activity and inversely proportional to the depth of the market.

It is immediate from (3.8) and the foregoing discussion that the tracking-error is essentially determined by the time-average of $u_{SS}^{\mathrm{BS}}(t,S_t)$ (the "Gamma" of the portfolio) along the future path of the stock price. While this confirms the practitioner's intuition that model-risk is essentially related to the Gamma of a portfolio, a word of warning is in order: $u_{SS}^{\mathrm{BS}}(0,S_0)$, i.e. the current Gamma tells us little about future values of Gamma and hence about the future model risk. A prime example for this is the case of an out-of-the-money put where the current Gamma is small but where the Gamma can become very large if the stock price drops to values around the strike price of the option shortly before the contract matures.

[4]One way to mitigate this problem is to use a volatility different from σ in computing the hedge-portfolio. This approach is studied in detail in Frey and Stremme (1997).

4 Nonlinear Black-Scholes equations

4.1 Perfect replication of derivatives

Consider a path-independent derivative with smooth payoff $h(S_T)$. It is natural to ask, if our trader can replicate a the payoff of such a derivative in our model where markets are not perfectly liquid, using a hedging strategy different from the standard Black-Scholes strategy.

Remark 4.1. If it exists, such a strategy is related to a fixed point of the volatility function $v(t,S)$ in the following sense: In a liquid market one can compute for a given stock-price volatility $v(t,S)$ a replicating strategy $\phi(t,S)$ using standard Black-Scholes theory. As shown in Proposition 3.1, if the large trader uses this strategy the actual volatility is transformed to $\tilde{v}(t,S) := 1/(1-\rho S\phi(t,S))$. Perfect hedging now requires that the volatility remains invariant under this transformation, i.e. that $v(t,S) \equiv \tilde{v}(t,S)$.

The following Proposition characterizes perfect replicating strategies in terms of a nonlinear Black-Scholes equation.

Proposition 4.2. *Assume that there is a solution $u \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^+)$ of the following nonlinear Black-Scholes equation*

$$u_t(t,S) + \frac{1}{2}\frac{\sigma^2}{(1-\rho S u_{SS}(t,S))^2} S^2 u_{SS}(t,S) = 0,\ u(T,S) = h(S)\,, \tag{4.1}$$

whose space derivative $u_S(t,S) := \frac{\partial u(t,S)}{\partial S}$ satisfies Assumption A5. Then the selffinancing strategy with stock-trading-strategy $\alpha_t = u_S(t,S_t)$ and value process $V_t = u(t,S_t)$, $0 \le t \le T$ is a perfect replication strategy for the derivative with payoff $h(S_T)$, i.e. the tracking-error e_T^M of this strategy is equal to zero.

The proof is given in Appendix A.3. A characterization of option-replicating strategies for a large trader in terms of a nonlinear PDE has previously been obtained in a number of papers including Frey (1998), Sircar and Papanicolaou (1998) and Schönbucher and Willmott (1998). We do not discuss existence and uniqueness of solutions to the terminal value problem (4.1) in this paper.[5]

With reference to Proposition 4.2 we will call the solution $u(t,S)$ hedge cost of the claim with payoff $h(S_T)$. Obviously the nonlinear PDE (4.1) cannot be solved explicitely so that we must resort to numerical techniques. Simulations for the hedge cost $u(t,S)$ and for the hedge-ratio $u_S(t,S)$, which were computed using the method of implicit finite differences, are are presented in Appendix B.

In Figure 1 we plot the hedge cost of a European Call with strike $K = 1$ and time to maturity one year for various values of the market liquidity ranging from $\rho = 0$ up to $\rho = 0.2$. It is obvious that the hedge-cost is increased with increasing values of ρ. This increase is most pronounced for $S \approx K$, which is due to the fact that the increase in volatility caused by dynamic hedging is most pronounced for these values of the stock price. In Figure 2 we give the corresponding values for the hedge ratio $\phi(1,S) := u_S(1,S)$. We see that the hedge-ratio flattens out with increasing values of ρ. This behaviour is typical for *convex* payoffs like options.

[5] This very technical issue is dealt with in Frey (1998) in a slightly different context. The results of this paper guarantee existence (for small ρ) and uniqueness for the terminal value problem (4.1).

Nonlinear Black-Scholes equations in Finance

Nonlinear PDE's which are very similar to the nonlinear Black-Scholes equation (4.1) arise frequently in the analysis of option hedging in incomplete markets or in markets with transaction costs, which makes these equations interesting from a risk-management viewpoint. To see this relation more clearly we generalize (4.1) slightly and consider nonlinear PDEs of the form

$$u_t(t,S) + \frac{1}{2}(v(t,S,u_{SS}))^2 S^2 u_{SS}(t,S) = 0 \tag{4.2}$$

for some function $v(t,S,q)$ which is *increasing* in q. The key feature of this PDE is of course the dependence of the "volatility" $v(t,S,q)$ on the second derivative of the solution (the "Gamma). We briefly mention some important models, where the hedge-cost of derivatives can be characterized by PDE's of the form (4.2):

- Incomplete markets and uncertain volatility: The nonlinear PDE proposed by Avellaneda, Levy, and Paras (1995) and Lyons (1995) for the computation of superhedging strategies in models with uncertain but bounded volatility is a special case of (4.2). In their model we have

  $$v(t,S,q) = \underline{\sigma} 1_{\{q \le 0\}} + \overline{\sigma} 1_{\{q > 0\}}\,,$$

  where $\underline{\sigma}$ and $\overline{\sigma}$ represent a lower and upper a-priori bound on the otherwise unspecified asset price volatility.

- Transaction costs: Leland (1985), Hoggard, Whalley, and Wilmott (1994) and in particular Barles and Soner (1998) have given asymptotic results which provide a characterization of the replication cost for a derivative in a Black-Scholes model with proportional transaction costs in terms of nonlinear PDE's which are all of the form (4.2) for an appropriate function v.

This indicates that PDE's of the form (4.2) are a useful tool in studying the model-risk that stems from the unrealistic assumptions of liquid, frictionless and complete markets underlying the standard Black-Scholes model; it also gives a precise mathematical interpretation to practitioner's intuition that model-risk in hedging portfolios of derivatives is to a large extent related to the degree of nonlinearity (as measured by Gamma) of the portfolio at hand.

5 Conclusion

In this paper we have studied the hedging of derivatives in a market which is not perfectly liquid. We considered a model where the implementation of a hedging strategy affects the price of the underlying asset. We found that in our setting the stock-price volatility depends on the hedging strategy of the large trader. We gave an explicit formula for the hedging error due to market illiquidity which allowed us to study the model risk related to the assumption of perfectly liquid markets in standard derivative asset analysis. We found that model risk is essentially related to the nolinearity of the payoff at hand as measured by the second derivative of the value of the hedge portfolio. Finally we gave a characterization of perfect hedging strategies in terms of a nonlinear Black-Scholes equation.

To conclude we mention some questions for further research. It would be interesting to study the consequences of our approach for the pricing of derivatives. In particular, market illiquidity might help to explain some of the pricing biases such as the skew pattern of implied volatility. Moreover, our results could be used to assess the cost of market illiquidity in hedging derivatives such that form appropriate reserves can be formed. From a theoretical viewpoint it would be highly desirable to come up with a better economic understanding of the determinants of market illiquidity which might than be used to refine the modelling of the present paper.

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A Proofs

A.1 Proof of Proposition 3.1

Itô's formula and Assumption A5 imply that the stockholdings α are a semimartingale. Again by Itô's formula we get from (3.2)

$$d\alpha_t = \phi_S(t,S_t)dS_t + \left(\phi_t(s,S_s) + \frac{1}{2}\phi_{SS}(t,S_t)v^2(t,S_t)(S_s)^2\right) ds\,. \tag{A.1}$$

Assumption A4 together with the equations (3.2) and (A.1) now yields the following dynamics for the equilibrium stock price process S

$$dS_t = \sigma S_t dW_t + \rho S_t \phi_S(t,S_t)dS_t + \rho S_t \left(\phi_t(t,S_t)dt + \frac{1}{2}\phi_{SS}(t,S_t)d\langle S\rangle_t\right)\,, \tag{A.2}$$

or, equivalently

$$(1-\rho S_t\phi_S(t,S_t))dS_t = \sigma S_t dW_t + \rho S_t\left(\phi_t(t,S_t)dt + \frac{1}{2}\phi_{SS}(t,S_t)d\langle S\rangle_t\right). \tag{A.3}$$

Under Assumption A5 the expression $(1-\rho S_t\phi_S(t,S_t))$ is strictly positive. Integrating $1/(1-\rho S_t\phi_S(t,S_t))$ over both sides of (A.3) therefore yields the following explicit form for the equilibrium stock price dynamics

$$dS_t = \frac{\sigma}{1-\rho S_t\phi_S(t,S_t)}S_t dW_t + \frac{\rho S_t}{1-\rho S_t\phi_S(t,S_t)}\left(\phi_t(t,S_t) + \frac{\sigma^2 S_t^2}{(1-\rho S_t\phi_S(t,S_t))^2}\right)dt\,,$$

which proves the Proposition. $\square$

A.2 Proof of Proposition 3.2

By definition we have

$$e_T^M = h(S_T) - \left(u^{\mathrm{BS}}(0,S_0) + \int_0^T u_S^{\mathrm{BS}}(t,S_t)dt\right). \tag{A.4}$$

If the large trader uses the standard Black-Scholes strategy (3.6), by Proposition 3.1 the stock price volatility equals $v(t,S;u^{\mathrm{BS}}) = \sigma/(1-\rho S u_{SS}^{\mathrm{BS}}(t,S))$. Applying Itô's formula to u^{BS} we get

$$\begin{aligned} h(S_T) = u^{\mathrm{BS}}(T,S_T) = u^{\mathrm{BS}}(0,S_0) &+ \int_0^T u_S^{\mathrm{BS}}(t,S_t)dS_t \\ &+ \int_0^T u_t^{\mathrm{BS}}(t,S_t) + \frac{1}{2}u_{SS}^{\mathrm{BS}}(t,S_t)v^2(t,S_t;u^{\mathrm{BS}})S_t^2 dt\,. \end{aligned}$$

Substituting this equation into the formula (A.4) we see that the stochastic integrals with respect to S cancel. If we now use the standard Black-Scholes PDE (3.5) to replace u_t^{BS} in the last line we obtain the desired formula (3.7). $\square$

A.3 Proof of Proposition 4.2

The proof uses the fixed-point argument outlined in Remark 4.1. If the large trader uses a stock-trading-strategy with $\alpha_t = u_S(t,S_t)$, the asset price volatility equals $v(t,S;u) = \sigma/(1-\rho S u_{SS}(t,S))$. Applying Itô's formula to u we get

$$\begin{aligned} h(S_T) = u(T,S_T) = u(0,S_0) &+ \int_0^T u_S(t,S_t)dS_t \\ &+ \int_0^T u_t(t,S_t) + \frac{1}{2}u_{SS}(t,S_t)v(t,S_t;u)^2 S_t^2 dt\,, \end{aligned}$$

where S stands for $S(\rho,\alpha)$. Now note that the last integral on the right vanishes because of (4.1). Hence we have the representation

$$h(S_T(\rho,\alpha)) = V_0 + \int_0^T \alpha_t dS_t(\rho,\alpha)$$

which shows that the tracking-error $e_T^M = 0$. $\square$

B Simulations

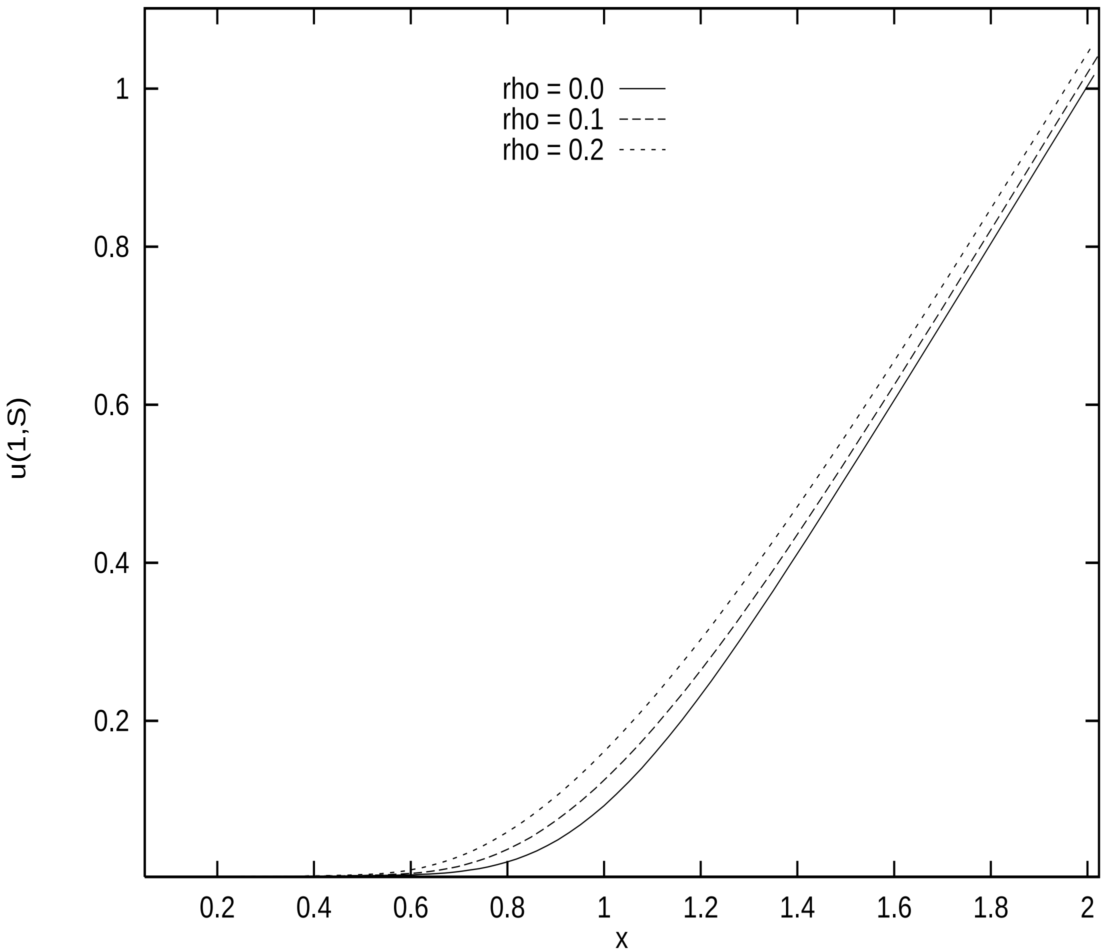

Figure 1: Hedge cost $u(t, S)$ for a European call with strike $K = 1$ and time to maturity one year for various values of ρ.

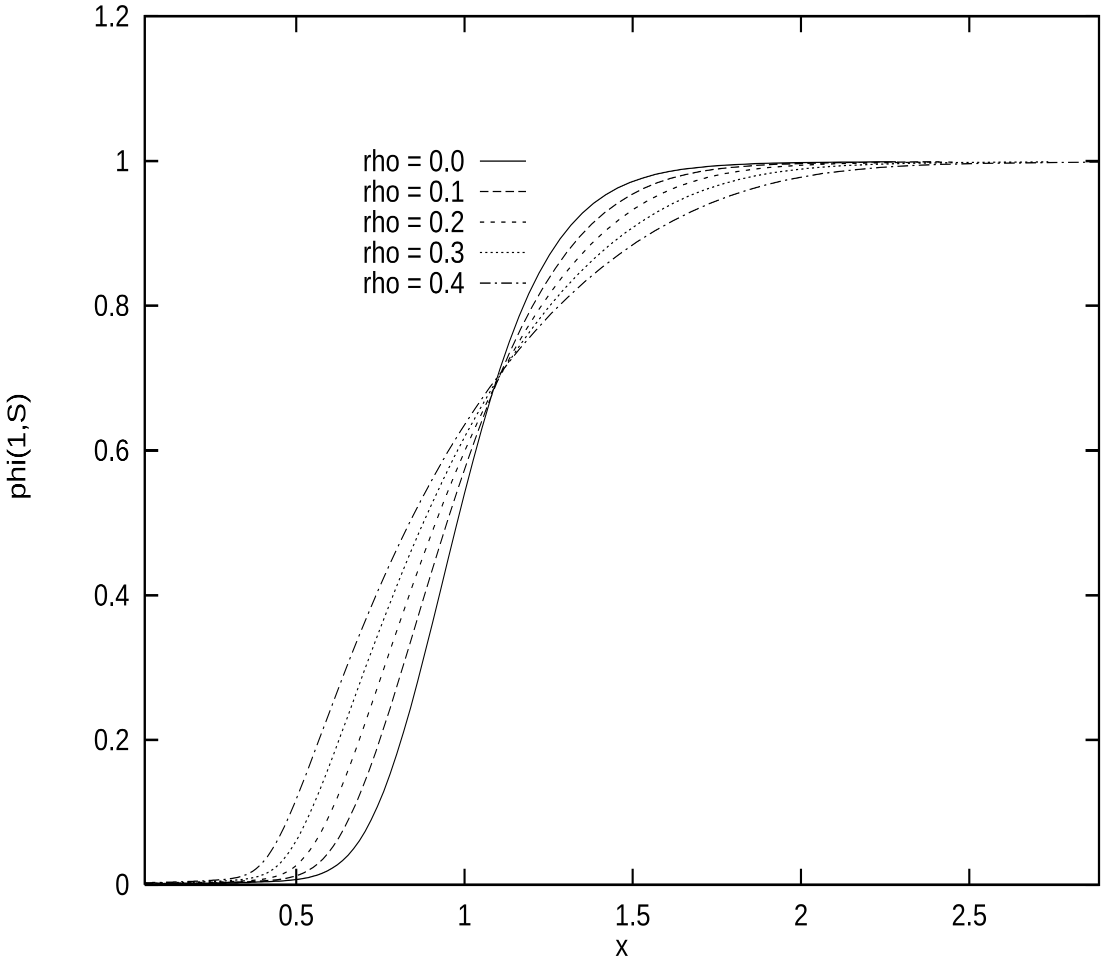

Figure 2: Hedge ratio $\phi(1, S) := u_S(1, S)$ for a European call with strike $K = 1$ and time to maturity one year for various values of ρ.