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Chapter 3

PRICING OPTIONS IN ILLIQUID MARKETS: SYMMETRY REDUCTIONS AND EXACT SOLUTIONS

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Abstract

This chapter is concerned with nonlinear Black-Scholes equations arising in certain option pricing models with a large trader and/or transaction costs. In the first part we give an overview of existing option pricing models with frictions. While the financial setup differs between models, it turns out that in many of these models derivative prices can be characterized by fully nonlinear versions of the standard parabolic Black-Scholes equation. In the second part of this chapter we study a typical nonlinear Black-Scholes equation using methods from Lie group analysis. The equation possesses a rich symmetry group. By introducing invariant variables, invariant solutions can therefore be characterized in terms of solutions to ordinary differential equations. Finally, we discuss properties and applications of these solutions.

1. Introduction

Standard derivative pricing theory is based on the assumption of frictionless markets. In particular, it is assumed that there are no transaction costs and that all investors are *small* relative to the market so that they can buy arbitrarily large quantities of the underlying assets without affecting its price (perfectly liquid or elastic markets). Given the scale of hedging activities on many financial markets this is clearly unrealistic. Hence in recent years a number of models for studying the pricing and the hedging of derivative securities in illiquid markets or in the presence of transaction costs have been developed.[1] The financial framework that is being used differs substantially between model classes. However, as shown below, in many of these models derivative prices can be characterized by fully nonlinear versions of the standard parabolic Black-Scholes equation; moreover, these nonlinear

[1]We discuss the relevant literature in the body of this chapter.

Black-Scholes equations have a very similar structure. This makes these equations a useful reference point for studying derivative asset analysis in illiquid markets.

The nonlinear Black-Scholes equations derived in the literature contain a parameter representing the size of the transaction costs or of the trader's impact on the market. Usually it is expected that the equations reduce to the linear Black-Scholes PDE if this parameter becomes small; put differently, the nonlinear equations are considered as *perturbations* of the linear Black-Scholes equation. From an analytical point of view these equations can be divided roughly into two classes: equations with regular perturbations and equations with singular perturbations. Equations of the latter type are fully nonlinear, and there are so far relatively few papers where equations with singular perturbations are studied by analytical methods. On the other hand, an analytical study of these nonlinear equations makes it possible to determine the scope of applications of different models represented by fully nonlinear partial differential equations (PDEs).

In this chapter we therefore apply Lie group analysis to a typical nonlinear Black-Scholes equation. It turns out that the equation possesses a rich Lie symmetry group which allows us to introduce invariant variables and to reduce the corresponding PDEs to ordinary differential equations (ODEs). It is even possible to find exact invariant solutions to the ensuing ODEs. We show that most of the exact solutions for a given nonlinear equation have no counterpart in the linear Black-Scholes case; they intrinsically reflect a nonlinearity of the equation. The last part of this chapter is devoted to applications. We study properties of solutions and discuss the sensitivity with respect to model parameters; in particular, we show that some solutions approximate typical financial derivatives relatively closely.

This chapter is organized as follows: the overview of existing option pricing models with limited market liquidity can be found in Section 2.; the analytic properties of nonlinear Black-Scholes equations are studied in Section 3; applications are discussed in Section 4.

2. Illiquid Markets and Nonlinear Black-Scholes Equations

In order to motivate the subsequent analysis we present a brief synopsis of three different frameworks for modeling illiquid markets. We group them under the labels *quadratic transaction-cost models*; *reduced-form SDE-models*; *reaction-function* or *equilibrium models*. In particular, we show that the value function of a certain type of self-financing strategies (so called *Markovian* strategies) must be a solution of a fully nonlinear version of the standard Black-Scholes equation. In all models there will be two assets, a risk-free money-market account B which is perfectly liquid and a risky and illiquid asset S (the stock), modelled on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Without loss of generality, we use the money market account as numeraire; hence $B_t \equiv 1$, and interest rates can be taken equal to zero.

2.1. (Quadratic) Transaction-Cost Models

The predominant model in this class has been put forward by Cetin, Jarrow and Protter [7], [8]. In this model there is a *fundamental* stock price process S^0 following geometric Brownian motion,

$$dS_t^0 = \mu S_t^0 dt + \sigma S_t^0 dW_t \tag{1}$$

for constants $\mu \in \mathbb{R}$, $\sigma > 0$ and a standard Brownian motion W. The *transaction price* to be paid at time t for trading α shares is

$$\bar{S}_t(\alpha) = e^{\rho\alpha} S_t^0, \quad \alpha \in \mathbb{R}, \;\; \rho > 0, \tag{2}$$

where ρ is a liquidity parameter.

Intuitively, in the model (2) a trader has to pay a spread whose size relative to the fundamental price equals $S_t^0(e^{\rho\alpha} - 1)$, so that the spread depends on the amount α to be traded. As shown in [7], this leads to transaction costs which are proportional to the *quadratic variation* of the stock trading strategy. In order to explain this statement in more detail, we consider a self-financing trading strategy $(\Phi_t, \eta_t)_{t \geq 0}$ giving the number of stocks and the position in the money market for predictable stochastic processes Φ and η. The *value* of this strategy at time t equals $V_t = \Phi_t S_t^0 + \eta_t$. Note that V_t is the so-called *paper value* of the position; under (2) the liquidation value of the strategy (the amount of money the large trader receives if he actually liquidates his stock position) is typically lower than V_t. For a detailed discussion of this point we refer to Bank and Baum [1].

In order to motivate the form of the dynamics of the paper value we consider a simple predictable strategy of the form $\Phi_t(\omega) = \sum_i \phi_i(\omega) 1_{(t_i, t_{i+1}]}(t)$ for deterministic time points $0 = t_0 < t_1 < \ldots$. Then the self-financing condition in t_i reads

$$\begin{aligned}
\eta_{t_i} - \eta_{t_{i-1}} &= -(\Phi_{t_i} - \Phi_{t_{i-1}})\, \bar{S}_{t_i}(\Phi_{t_i} - \Phi_{t_{i-1}}) \\
&= -\left(\Phi_{t_i} - \Phi_{t_{i-1}}\right)\left(S_{t_i}^0 + \rho S_{t_i}^0(\Phi_{t_i} - \Phi_{t_{i-1}})\right) + o\left((\Phi_{t_i} - \Phi_{t_{i-1}})^2\right) \,.
\end{aligned}$$

Note that in the last line we have used that $\frac{d}{d\alpha}_{|\alpha=0} \bar{S}_t(\alpha) = S_t^0$. Hence the change in the value of the portfolio equals

$$\begin{aligned}
V_{t_i} - V_{t_{i-1}} &= \Phi_{t_i} S_{t_{i+1}}^0 - \Phi_{t_{i-1}} S_{t_i}^0 + \eta_{t_i} - \eta_{t_{i-1}} \\
&= \Phi_{t_i}(S_{t_{i+1}}^0 - S_{t_i}^0) - \rho S_{t_i}^0 \left(\Phi_{t_i} - \Phi_{t_{i-1}}\right)^2 + o\left((\Phi_{t_i} - \Phi_{t_{i-1}})^2\right) \,.
\end{aligned}$$

This suggests that for a continuous semi-martingale Φ with quadratic variation[2] $[\Phi]_t$ the wealth dynamics of a self-financing strategy becomes

$$dV_t = \Phi_t dS_t^0 - \rho S_t^0 d[\Phi]_t. \tag{3}$$

This is in fact true as is shown in Theorem A3 of [7]; see also [1]. Note that the last term in (3) represents the extra transaction cost due to the limited liquidity of the market.

Consider now a *Markovian strategy*, that is a trading strategy of the form $\Phi_t = \phi(t, S_t^0)$ for a smooth function ϕ. In this case Φ is a semi-martingale with quadratic variation given by

$$[\Phi]_t = \int_0^t \left(\phi_S(s, S_s^0)\sigma S_s^0\right)^2 ds \,;$$

see for instance Chapter 2 of [29]. Combining this with (3) yields the following dynamics of V_t

$$dV_t = \phi(t, S_t^0) dS_t^0 - \rho S_t^0 \left(\phi_S(t, S_t^0)\sigma S_t^0\right)^2 dt. \tag{4}$$

[2]See for instance Chapter 2 of [29] for a definition of quadratic variation.

Suppose now that u and ϕ are smooth functions such that $u(t, S_t^0)$ gives the value of a self-financing trading strategy with stock position $\phi(t, S_t^0)$. According to the Itô formula, the process $(u(t, S_t^0))_{t \geq 0}$ has dynamics

$$du(t, S_t^0) = u_S(t, S_t^0) dS_t^0 + \left(u_t(t, S_t^0) + \frac{1}{2}\sigma^2 (S_t^0)^2 u_{SS}(t, S_t^0) \right) dt \,. \tag{5}$$

By the uniqueness of semi-martingale decompositions it is immediate that the dt-terms in (4) and (5) have to coincide, so that u must satisfy the equation $u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} + \rho S^3 \sigma^2 \phi_S = 0$. Moreover, we must have that $\phi \equiv u_S$. The last relation gives $\phi_S = u_{SS}$, so that we obtain the following nonlinear PDE for u

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} \left(1 + 2\rho S u_{SS}\right) = 0 \,. \tag{6}$$

When pricing a derivative security with maturity date T and payoff $h(S_T)$ for some function $h : [0, \infty) \to \mathbb{R}$ we have to add the terminal condition $u(T, S) = h(S)$, $S \geq 0$. For instance, in case of a European call option with strike price $K > 0$ we have $h(S) = \max\{S - K, 0\}$.

Note that the original paper [7] goes further in the analysis of quadratic transaction cost models. To begin with, a general framework is proposed that contains (2) as special (but typical) case. Moreover, conditions for absence of arbitrage for a general class of trading strategies - containing Markovian trading strategies as special case - are given, and a notion of approximative market completeness is studied.

2.2. Reduced-Form SDE Models

Under this modeling approach it is assumed that investors are *large traders* in the sense that their trading activity affects equilibrium stock prices. More precisely, given a liquidity parameter $\rho \geq 0$ and a semi-martingale Φ representing the stock trading strategy of a given trader, it is assumed that the stock price satisfies the stochastic differential equation (SDE)

$$dS_t = \sigma S_t dW_t + \rho S_t d\Phi_t \,. \tag{7}$$

The intuitive interpretation is as follows: given that the investor buys (sells) stock ($\Delta\Phi_t > 0$) the stock price is pushed (downward) upward by $\rho S_{t-} \Delta\Phi_t$; the strength of this price impact depends on the parameter ρ. Note that for $\rho = 0$ the asset price simply follows a Black-Scholes model with reference volatility σ. The model (7) and slight variants thereof are studied among others in [11], [12], [18], [23], [28] or [14].

In the sequel we denote the asset price process which results if a large trader uses a particular trading strategy Φ by S^Φ. Suppose as before that the trading strategy is Markovian, i.e. of the form $\Phi_t = \phi(t, S_t)$ for a smooth function ϕ and that ϕ satisfies the constraint

$$1 - \rho S \phi_S(t, S) > 0 \text{ for all } (t, S);$$

given a liquidity parameter ρ, this last condition basically limits the permissible variations in the stock trading strategy of the large trader. Applying the Itô formula to (7) shows that S^ϕ is an Itô process with dynamics

$$dS_t^\phi = v^\phi(t, S_t^\phi) S_t^\phi dW_t + b^\phi(t, S_t^\phi) S_t^\phi dt$$

with *adjusted volatility* given by

$$v^{\phi}(t,S) = \frac{\sigma}{1-\rho S \phi_S(t,S)}\,; \tag{8}$$

see for instance [11] for a detailed derivation. Note that this adjusted volatility is increased (decreased) relative to the reference volatility σ if $\phi_S > 0$ ($\phi_S < 0$).

In the model (7), a portfolio with stock trading strategy Φ and value V is termed *self-financing*, if satisfies the equation $dV_t = \Phi_t dS_t^{\Phi}$. Note that the form of the strategy Φ affects the dynamics of S; this *feedback effect* will give rise to nonlinearities in the wealth dynamics as we now show. Suppose that $V_t = u(t,S_t)$ and $\Phi_t = \phi(t,S_t)$ for smooth functions u and ϕ. As before, applying the Itô formula to the process $(u(t,S_t^{\phi}))_{t\geq 0}$ yields $\phi \equiv u_S$. Moreover, u must satisfy the relation $u_t + \frac{1}{2}(v^{\phi})^2(t,S)S^2 u_{SS} = 0$. Using (8) and the relation $\phi_S = u_{SS}$ we thus obtain the following fully nonlinear PDE for $u(t,S)$

$$u_t + \frac{1}{2}\frac{\sigma^2}{(1-\rho S u_{SS})^2} S^2 u_{SS} = 0\,. \tag{9}$$

Again, for pricing derivative securities a terminal condition corresponding to the particular payoff at hand needs to be added.

2.3. Equilibrium or Reaction-Function Models

Here the model primitive is a smooth *reaction function* ψ that gives the equilibrium stock price S_t at time t as function of some fundamental value F_t and the stock position of a large trader. A reaction function can be seen as reduced-form representation of an economic equilibrium model, such as the models proposed in [13], [27] or [30]. In these models there are two types of traders in the market: ordinary investors and a large investor. The overall supply of the stock is normalized to one. The normalized stock demand of the ordinary investors at time t is modelled as a function $D(F_t, S_t)$ where S_t is the proposed price of the stock. The normalized stock demand of the large investor is written the form $\rho\Phi_t$; $\rho \geq 0$ is a parameter that measures the size of the trader's position relative to the total supply of the stock. The equilibrium price S_t is then determined by the market clearing condition

$$D(F_t, S_t) + \rho\Phi_t = 1\,. \tag{10}$$

Under suitable assumptions on D, equation (10) admits a unique solution. Hence S_t can be expressed as a function ψ of F_t and $\rho\Phi_t$, so that $S_t = \psi(F_t, \rho\Phi_t)$. For instance we have in [27] that $\psi(f,\alpha) = f\exp(\alpha)$; the model used in [13] and [32] leads to the reaction function $\psi(f,\alpha) = f/(1-\alpha)$. The reaction-function approach is also used in [19] and in [10].

Now we turn to the characterization of self-financing hedging strategies in reaction-function models. Throughout we assume that the fundamental-value process F follows a geometric Brownian motion with volatility σ as in (1). Moreover, we assume that the reaction function is of the form $\psi(f,\alpha) = fg(\alpha)$ for some increasing function g. This holds for the specific examples introduced above and, more generally, for any model where $D(f,s) = U(f/s)$ for a strictly increasing function $U\colon (0,\infty) \to \mathbb{R}$ with suitable range.

Assuming as before that the normalized trading strategy of the large trader is of the form $\rho\phi(t,S)$ for a smooth function ϕ, we get from Itô's formula that

$$dS_t = g(\rho\phi(t,S_t))\, dF_t + \rho F_t g_\alpha(\rho\phi(t,S_t))\phi_S(t,S_t)\, dS_t + b(t,S_t)\, dt \tag{11}$$

(since $S_t = g(\rho\phi(t,S_t))F_t$); the precise form of $b(t,S_t)$ is irrelevant for our purposes. Assume now that

$$\big(1 - \rho F_t g_\alpha(\rho\phi(t,S_t))\phi_S(t,S_t)\big) > 0 \quad \text{a.s.}; \tag{12}$$

as before this can be viewed as an upper bound on the permissible variations of the large trader's strategy. Rearranging and integrating $\big(1 - \rho F_t g_\alpha(\rho\phi(t,S_t))\phi_S(t,S_t)\big)^{-1}$ over both sides of equation (11) gives the following dynamics of S:

$$dS_t = \frac{1}{1 - \rho\frac{g_\alpha(\rho\phi(t,S_t))}{g(\rho\phi(t,S_t))} S_t\phi_S(t,S_t)}\sigma S_t dW_t + \tilde{b}(t,S_t)dt\,; \tag{13}$$

again the precise form of $\tilde{b}$ is irrelevant.

A similar reasoning as in the case of the reduced-form SDE models now gives the following PDE for the value function $u(t,S)$ of a self-financing strategy

$$u_t + \frac{1}{2}\frac{\sigma^2}{\Big(1 - \rho\frac{g_\alpha(\rho u_S)}{g(\rho u_S)} S u_{SS}\Big)^2} S^2 u_{SS} = 0\,. \tag{14}$$

In particular, for $g(\alpha) = \exp(\alpha)$ we have $g = g_\alpha$ and (14) reduces to equation (9); for $g(\alpha) = 1/(1-\alpha)$ as in [13], [32], we get the PDE

$$u_t + \frac{1}{2}\frac{\sigma^2(1-\rho u_S)^2}{(1-\rho u_S - \rho S u_{SS})^2} S^2 u_{SS} = 0\,. \tag{15}$$

A thorough analysis of the dynamics of self-financing strategies in generalized reaction-function models via the *Itô-Wentzell formula* can be found in [1].

2.4. Nonlinear Black-Scholes Equations

The nonlinear PDEs (6), (9), (14) and (15) are all of the form

$$u_t + \frac{1}{2}\sigma^2 S^2 v(\rho u_S, \rho S u_{SS}) u_{SS} = 0\,, \tag{16}$$

where $v(0,0) = 1$. Since ρ is often considered to be small, it is of interest to replace v with its first order Taylor approximation around $\rho = 0$. It is immediately seen that for the equations (9) and (15) this linearization is given by $v(\rho u_S, \rho S u_{SS}) \approx 1 + \rho S u_{SS}$; replacing $v(\rho u_S, \rho S u_{SS})$ with this first order Taylor approximation in (9) and (15) thus immediately leads to the PDE (6).

Note that (16) is a fully nonlinear equation in the sense that the coefficient of the highest derivative is a nonlinear function of this derivative. A similar feature can be observed for the limiting price in certain transaction cost models under a proper re-scaling of transaction cost and trading frequency; see for instance [2] or [16]. Nonlinear PDEs for incomplete markets obtained via exponential utility indifference hedging such as [3] or [24] on the other hand are quasi-linear equations in the sense that the highest derivative u_{SS} enters the equation in a linear way, similar to the well-known reaction-diffusion equations arising in physics or chemistry. From an analytical point of view the nonlinearities arising in both cases are quite different. In the latter quasi-linear case we have to do with a regular perturbation of the classical Black-Scholes (BS) equation but in the case (14) we have to do with a singular perturbation, meaning, in particularly, that the highest derivative is included in the perturbation [20]. In case of a regular perturbation it is typical to look for a representation of a solutions to a quasi-linear equation in the form

$$u\,(S,t) = u_{BS}(S,t) + \sum_{n=1}^{N} \rho u_1(S,t) + O(\rho^{N+1}). \tag{17}$$

For many forms of nonlinearities the uniform convergence as $\rho \to 0$ of the expansions of type (17) can be established. If the perturbation is of the singular type, the asymptotic expansion of the form (17) typically breaks down for some S, t and some $N \geq 0$. It can as well happens that the definition domain for the solutions of type (17) include just one point or that it is empty. It is very important to obtain explicit solutions for such models because in these case we can not hope to get good approximations for solutions by expansions of the type (17) in the whole region. This issue is taken up in the remainder of the chapter.

3. Invariant Solutions for a Nonlinear Black-Scholes Equation

We have seen in the previous section that the form of the equation (16) is typical for nonlinear PDEs arising in pricing equations for derivatives in illiquid markets. One of the methods to study properties of solutions to such type of fully nonlinear PDEs is the Lie group analysis of these equations. This approach can be applied to all equations listed before, i.e., to (6), (9) and (15). The first results were achieved for the equation (9). In [6] a special family of invariant solutions to the equation (9) was studied; in particular, the explicit solutions were used as test case for various numerical methods. Later on the slightly modified equation (32) was studied in [5] where the complete family of invariant solutions was described. The symmetry groups of the equations (6) and (15) were found in [4] and [31] correspondingly.

Our goal in this section is to investigate the nonlinear Black-Scholes equation (9) using analytical methods. Using the symmetry group and its invariants the PDE (9) can be reduced in special cases to ODEs. In the present chapter we study *all* invariant solutions to equation (9).

In the next paragraph 3.1. we provide a description of the Lie group method in application to the differential equations. This short introduction can by no means replace any study of classical books devoted to the method but it will allowed us to introduce necessary notations.

3.1. Lie Group Analysis of Differential Equations

In this paragraph we formulate all definitions for the case of one PDE of two independent and one dependent variables. It makes ideas transparent and easier accessible for somebody who does not used this method before. On the other side these formulations are general enough to study the equations of type (16). As a practical and easy introduction to this theory one can take as well the book [33]. The most general formulations can be found in books cited below.

Let us introduce a two-dimensional space X of independent variables $(S, t) \in X$ and a one-dimensional space of dependent variables $u \in U$. In our case X is isomorphic to a two-dimensional Euclidean space $\mathbb{R}^2$ and U to $\mathbb{R}$. We consider the space $U_{(1)}$ endowed with differential coordinates (u_S, u_t) which represent all the first derivatives of the variable u on S and t. Analogously, we introduce the space $U_{(2)}$ endowed with differential coordinates (u_{SS}, u_{St}, u_{tt}) which represent all the second order derivatives of the dependent variable u on S and t. We can continue with this procedure and introduce spaces of the type $U_{(n)}$, $n > 2$. As long we study an equation of the order two it will be sufficient to introduce $U_{(n)}$ up to the order $n = 2$.

We denote by M an open subset of the Cartesian product $M \subset X \times U$. The space M is an underlying space with elements denoted by (x, u) with $x = (S, t) \in X, \;\; u \in U$. We denote an element of a vector field on M by

$$V = \xi(S,t,u)\frac{\partial}{\partial S} + \tau(S,t,u)\frac{\partial}{\partial t} + \phi(S,t,u)\frac{\partial}{\partial u}, \tag{18}$$

where $\xi(S, t, u)$,$\tau(S, t, u)$ and $\phi(S, t, u)$ are smooth functions of their arguments, $V \in \mathcal{D}iff(M)$. The operators (18) are called as well ***infinitesimal generators***.

The differential equation (16) is of the second order and to represent this equation as an algebraic relation on an appropriate space we introduce a second order jet bundle $M^{(2)}$ of the space M, i.e.,

$$M^{(2)} = M \times U_{(1)} \times U_{(2)} \tag{19}$$

with a natural contact structure [15], [25]. We label the coordinates in the jet bundle $M^{(2)}$ by $w = (S, t, u, u_S, u_t, u_{SS}, u_{St}, u_{tt}) \in M^{(2)}$.

The corresponding vector fields on $M^{(2)}$ have the form

$$\begin{aligned} pr^{(2)}V = {} & \xi(S,t,u)\frac{\partial}{\partial S} + \tau(S,t,u)\frac{\partial}{\partial t} + \phi(S,t,u)\frac{\partial}{\partial u} \\ & + \phi^S(S,t,u)\frac{\partial}{\partial u_S} + \phi^t(S,t,u)\frac{\partial}{\partial u_t} \\ & + \phi^{SS}(S,t,u)\frac{\partial}{\partial u_{SS}} + \phi^{St}(S,t,u)\frac{\partial}{\partial u_{St}} + \phi^{tt}(S,t,u)\frac{\partial}{\partial u_{tt}}. \end{aligned} \tag{20}$$

The vector fields $pr^{(2)}V$ are called the second prolongation of the vector fields V. Here the smooth functions $\phi^S(S, t, u)$, $\phi^t(S, t, u)$, $\phi^{SS}(S, t, u)$, $\phi^{St}(S, t, u)$ and $\phi^{tt}(S, t, u)$ are uniquely defined by the functions $\xi(S, t, u), \tau(S, t, u)$ and $\phi(S, t, u)$ (18) using the prolongation procedure (see [26], [25], [33], [15], [17]).

Remark 3.1. In the very simplified way we can present the idea behind this prolongation procedure as follows. We introduce some equivalence relation on the space of smooth functions defined on some open subset of X. Let a point x belong to this open subset. We call the functions equivalent in the point x if all their derivatives up to the order n coincide in the point $x \in X$. We call this equivalence class attached to the point x $n-$jet. It means at the same time as we introduced jets we introduced on the space a natural contact structure and prolonged our space by $U_{(n)},\ n \geq 1$.

Because of the very special form of the equation (9) we will use in our further calculations the exact form of the coefficients $\phi^t(S,t,u)$ and $\phi^{SS}(S,t,u)$ only. The coefficient $\phi^t(S,t,u)$ can be defined by the formula

$$\phi^t(S,t,u) = \phi_t + u_t\phi_u - u_S\xi_t - u_S u_t\xi_u - u_t\tau_t - (u_t)^2\tau_u, \tag{21}$$

and the coefficient $\phi^{SS}(S,t,u)$ by the expression

$$\begin{aligned}\phi^{SS}(S,t,u) = {} & \phi_{SS} + 2u_S\phi_{Su} + u_{SS}\phi_u \\ & + (u_S)^2\phi_{uu} - 2u_{SS}\xi_S - u_S\xi_{SS} - 2(u_S)^2\xi_{Su} \\ & - 3u_S u_{SS}\xi_u - (u_S)^3\xi_{uu} - 2u_{St}\tau_S - u_t\tau_{SS} \\ & - 2u_S u_t\tau_{Su} - (u_t u_{SS} + 2u_S u_{St})\tau_u - (u_S)^2 u_t\tau_{uu},\end{aligned} \tag{22}$$

where the subscripts by ξ, τ, ϕ denote corresponding partial derivatives. If we would study equation of the type (16) then we would need as well the form of the coefficient $\phi^S(S,t,u)$ which has a similar structure to the listed above.

Remark 3.2. The jet bundle $M^{(2)}$ is an example of a locally trivial smooth vector bundle $(M^{(2)}, \pi, X)$ where π is a smooth map $\pi \to X$ and X is the base space (see [21]).

In the jet bundle $M^{(2)}$ equation (16) is equivalent to the relation

$$\Delta(w) = 0, \ \ w \in M^{(2)}, \tag{23}$$

where we denote by Δ the following function

$$\Delta(S,t,u,u_S,u_t,u_{SS},u_{St},u_{tt}) = u_t + \frac{1}{2}\sigma^2 S^2 v(\rho u_S, \rho S u_{SS})u_{SS}. \tag{24}$$

We identify the algebraic relation (23) with its solution manifold $L_\triangle$ defined by

$$L_\triangle = \{w \in M^{(2)} | \Delta(w) = 0\} \subset M^{(2)}. \tag{25}$$

Let us consider an action of a Lie-point group on our differential equation and its solutions. We define a symmetry group $G_\triangle$ of equation (23) by

$$G_\triangle = \{g \in \mathrm{Diff}(\mathrm{M}^{(2)}) | \ \ \mathrm{g}: \ \ \mathrm{L}_\triangle \to \mathrm{L}_\triangle\}, \tag{26}$$

consequently we are interested in a subgroup of $\mathrm{Diff}(\mathrm{M}^{(2)})$ which is compatible with the structure of $L_\triangle$.

We follow as usual the idea of Sophus Lie [22]: instead to determine directly a complicated structure of the Lie-point symmetry group $G_\triangle$ we first determine the corresponding symmetry Lie algebra $\mathcal{D}iff_\triangle(M^{(2)}) \subset \mathcal{D}iff(M^{(2)})$ and then use the main Lie theorem to obtain $G_\triangle$ and its invariants.

The symmetry algebra $\mathcal{D}iff_\Delta(M^{(2)})$ of the second order differential equation (23) can be found as a solution to the determining equations

$$pr^{(2)}V(\Delta) = 0 \; (mod(\Delta = 0)), \tag{27}$$

the expression $mod(\Delta = 0)$ means that the equation (27) should be satisfied on the solution manifold L_Δ only.

If the infinitesimal generators $pr^{(2)}V \in \mathcal{D}iff_\Delta(M^{(2)})$ exist then they have the structure of type (20) and form a Lie algebra $\mathcal{D}iff_\Delta(M^{(2)})$. All of these infinitesimal generators are uniquely defined as prolongations of operators of the type (18). It is well known [22], [26], [25] that the prolongation procedure preserves the Lie algebraic structure and we can take into account for further studies the algebra $\mathcal{D}iff_\Delta(M)$ only. The algebra $\mathcal{D}iff_\Delta(M^{(2)})$ has the same algebraic structure as $\mathcal{D}iff_\Delta(M)$.

The symmetry algebra $\mathcal{D}iff_\Delta(M)$ defines by the Lie equations the corresponding symmetry group G_Δ of the equation (23)(see [22], [26], [25]). The Lie group of transformations $G_\Delta(M)$ acting on the space M induce in unique manner the transformations on $M^{(2)}$ which form a group denoted by G_Δ.

To find the global form of point transformations for the solutions to equation (16) corresponding to this symmetry group $G_\Delta(M)$ we just integrate the system of ODEs. The Lie equations in our case take a form

$$\frac{d\tilde{S}}{d\epsilon} = \xi(\tilde{S}, \tilde{t}, u), \tag{28}$$

$$\frac{d\tilde{t}}{d\epsilon} = \tau(\tilde{S}, \tilde{t}, \tilde{u}), \tag{29}$$

$$\frac{d\tilde{u}}{d\epsilon} = \phi(\tilde{S}, \tilde{t}, \tilde{u}), \tag{30}$$

with initial conditions

$$\tilde{S}|_{\epsilon=0} = S, \quad \tilde{t}|_{\epsilon=0} = t, \quad \tilde{u}|_{\epsilon=0} = u. \tag{31}$$

Here the variables $\tilde{S}, \tilde{t}$ and $\tilde{u}$ denote values S, t, u after a symmetry transformation. The parameter ϵ describes a motion along an orbit of the group. Usually it is possible to solve the autonomous system of equations (28) in closed analytical form.

If we have global transformations which form the symmetry group $G_\Delta(M)$ it is rather easy to obtain analytical expressions which are invariant under the action of this Lie group, i.e. invariants of the group. One can obtain invariants as well on a direct way by solving a system of equations which follows from the Lie equations. Both methods are equivalent and provide the same set of invariants. The form of invariants is not unique because any function of invariants is as well an invariant. Always we obtain one and the same number of functionally independent invariants. In case of ODEs invariants are the first integrals and if we have a rich set of invariants with some additional properties then we can obtain solutions to the ODEs without any additional integration procedure. In case of PDEs the situation is much more complicated and we cannot hope to obtain in some sense general solutions to a PDE. Using the invariants we can introduce invariant variables and reduce the partial differential equation to some set of ODEs. Each of these ODEs gives rise to a family of group-invariant solutions to the partial differential equation. The famous Black-Scholes formula for the Call option is one of the examples for the invariant solutions of these type.

3.2. Previous Results

In this paragraph we give a short overview of the results presented in [5]. These results will be used in the next paragraph to provide the complete set of invariant solutions to the equation (9).

In the paper [5] the slightly more general equation

$$u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{(1-\rho\lambda(S) S u_{SS})^2} = 0, \tag{32}$$

with a continuous function $\lambda : (0,\infty) \to (0,\infty)$ is studied (note that for $\lambda \equiv 1$ this equation reduces to (9)) and the following theorems were proved.

Theorem 3.1. *The differential equation (32) with an arbitrary function $\lambda(S)$ possesses a trivial three dimensional Lie algebra $Diff_\Delta(M)$ spanned by infinitesimal generators*

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = S\frac{\partial}{\partial u}, \quad V_3 = \frac{\partial}{\partial u}.$$

Only for the special form of the function $\lambda(S) \equiv \omega S^k$, where $\omega, k \in \mathbb{R}$ equation (32) *admits a non-trivial four dimensional Lie algebra spanned by generators*

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = S\frac{\partial}{\partial u}, \quad V_3 = \frac{\partial}{\partial u}, \quad V_4 = S\frac{\partial}{\partial S} + (1-k)u\frac{\partial}{\partial u}. \tag{33}$$

The proof of this theorem is rather easy. We solve the system of the determining equations (27) where Δ is defined by (32). The calculations are tedious but straightforward. Finally, we obtain the admissible form of the coefficients ξ, τ, ϕ in (18). We notice that the four dimensional Lie algebra (33) has a three dimensional Abelian subalgebra and in case $k = 0, 1$ a two dimensional center. In these two cases it is possible to find invariant solutions in closed analytical form.

On the next step we use the Lie algebra (33) to obtain the symmetry group of the studied equation. The Lie equations (28) in case of the Lie algebra (33) are simple. The results of these calculations are presented in the form of the following theorem.

Theorem 3.2. *The action of the symmetry group G_Δ of (32) with an arbitrary function $\lambda(S)$ is given by*

$$\tilde{S} = S, \tag{34}$$

$$\tilde{t} = t + \epsilon_1, \tag{35}$$

$$\tilde{u} = u + S\epsilon_2 + \epsilon_3, \qquad \epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}. \tag{36}$$

If the function $\lambda(S)$ has the special form $\lambda(S) = \omega S^k,\ k, \omega \in \mathbb{R}$, equation (32) takes the special form

$$u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{(1-bS^{k+1}u_{SS})^2} = 0, \quad b = \omega\rho. \tag{37}$$

In that case the symmetry group G_Δ has the richer structure

$$\begin{aligned}
\tilde{S} &= Se^{\epsilon_1}, && (38)\\
\tilde{t} &= t+\epsilon_2,\\
\tilde{u} &= ue^{(1-k)\epsilon_1}+\frac{\epsilon_3}{k}Se^{\epsilon_1}(1-e^{-k\epsilon_1})\\
&\quad +\frac{\epsilon_4}{(1-k)}(e^{(1-k)\epsilon_1}-1), \quad k\neq 0, k\neq 1, && (39)\\
\tilde{u} &= ue^{\epsilon_1}+S\epsilon_3 e^{\epsilon_1}+\epsilon_4(e^{\epsilon_1}-1), \quad k=0,\\
\tilde{u} &= u+\epsilon_3 S(e^{\epsilon_1}-1)+\epsilon_4, \quad k=1. && (40)\\
&\quad \epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\mathbb{R}.
\end{aligned}$$

As we will see later, the solutions found in [5] for the case $\lambda(S)=S$, i.e. for the case $k=1$ can be used to obtain the complete set of invariant solutions to equation (9), i.e. for the case $k=0$.

For $\lambda(S)=\omega S^k$ with $k=0,1$ we can use (38) to obtain a new independent invariant variable z and (40) to obtain a new dependent variable v; these variables are given by

$$\begin{aligned}
z &= \log S + at, \quad a\neq 0,\\
v &= u\, S^{(k-1)}, \quad k=0,1.
\end{aligned} \tag{41}$$

Using these variables we can reduce the PDE (37) to the ODE

$$v_z + q\frac{v_{zz}+\xi v_z}{(1-b(v_{zz}+\xi v_z))^2}=0, \tag{42}$$

where $q=\frac{\sigma^2}{2a}$, $a\neq 0$, $b=\omega\rho\neq 0$, $\xi=(-1)^k$, $k=0,1$. It is a straightforward consequence of Theorem 3.2 that these are the only nontrivial invariant variables. In the sequel we will determine explicit solutions for (42) for the case $k=0$ which corresponds to the original equation (9).

Remark 3.3. Equation (42) with $k=0$ and $k=1$ are related to each other. Note however, that the relation between the corresponding solutions is not so straightforward, because (42) is nonlinear and we need real valued solutions. Hence the results from [5] — where (42) was studied for the case $k=1$ — do not carry over directly to the present case $k=0$, so that a detailed analysis of the case $k=0$ is necessary.

3.3. Symmetry Reductions of the Equation (9)

We use the previous results and continue with an analytical treatment of the equation (42) in case $k=0$. To find families of invariant solution to (9) we introduce a new dependent variable

$$y(z)=v_z(z). \tag{43}$$

If we assume that the denominator of the equation (42) is different from zero, we can multiply both terms of equation (42) by the denominator of the second term and obtain

$$yy_z^2+2\xi\left(y^2-\xi\frac{1}{b}y+\xi\frac{q}{2b^2}\right)y_z+\left(y^2-\xi\frac{2}{b}y+\left(\frac{1+\xi q}{b^2}\right)\right)y=0, \quad b\neq 0. \tag{44}$$

We denote the left hand side of this equation by $F(y, y_z)$. The equation (44) can possess exceptional solutions which are the solutions of a system

$$\frac{\partial F(y, y_z)}{\partial y_z} = 0, \qquad F(y, y_z) = 0. \tag{45}$$

The first equation in this system defines a discriminant curve which has the form

$$y(z) = \frac{q}{4b}. \tag{46}$$

If this curve is also a solution of the original equation (44) then we obtain an exceptional solution to equation (44). We obtain an exceptional solution if $q = -4\xi$, i.e. $a = \xi\sigma^2/8$. It has the form

$$y(z) = -\frac{\xi}{b}, \quad \xi = (-1)^k, \quad k = 0, 1. \tag{47}$$

This solution belongs to the family of solutions (49) by the specified value of the parameter q. In all other cases the equation (44) does not possess any exceptional solutions.

Hence the set of solutions of equation (44) is the union of the solution sets of the following equations

$$y = 0, \tag{48}$$

$$y = \frac{\xi \pm \sqrt{q}}{b}, \tag{49}$$

$$y_z = \left(-\xi y^2 + \frac{1}{b}y - \frac{q}{2b^2} - \sqrt{\frac{q}{b^3}\left(\frac{q}{4b} - y\right)}\right)\frac{1}{y}, \quad y \neq 0, \tag{50}$$

$$y_z = \left(-\xi y^2 + \frac{1}{b}y - \frac{q}{2b^2} + \sqrt{\frac{q}{b^3}\left(\frac{q}{4b} - y\right)}\right)\frac{1}{y}, \quad y \neq 0, \tag{51}$$

where one of the solutions (49) is an exceptional solution (47) by $q = -4\xi$ for $k = 1$. In the case $k = 0$ we have $\xi = 1$ so that solutions of (49) are complex valued functions. We denote the right hand side of equations (50), (51) by $f(y)$. The Lipschitz condition for equations of the type $y_z = f(y)$ is satisfied in all points where the derivative $\frac{\partial f}{\partial y}$ exists and is bounded. It is easy to see that this condition will not be satisfied by

$$y = 0, \;\; y = \frac{q}{4b}, \;\; y = \infty. \tag{52}$$

Hence on the lines (52) the uniqueness of solutions of equations (50), (51) can be lost. We will study in detail the behavior of solutions in the neighborhood of the lines (52). For this purpose we look at the equation (44) from another point of view. If we assume now that z, y, y_z are complex variables and denote

$$y(z) = \zeta, \;\; y_z(z) = w, \;\; \zeta, w \in \mathbb{C}, \tag{53}$$

then the equation (44) takes the form

$$F(\zeta, w) = \zeta w^2 + 2\xi\left(\zeta^2 - \xi\frac{1}{b}\zeta + \xi\frac{q}{2b^2}\right)w + \left(\zeta^2 - \xi\frac{2}{b}\zeta + \frac{1 + \xi q}{b^2}\right)\zeta = 0, \tag{54}$$

where $b \neq 0$. The equation (54) is an algebraic relation in $\mathbb{C}^2$ and defines a plane curve in this space. The polynomial $F(\zeta, w)$ is an irreducible polynomial if at all roots $w_r(z)$ of $F(\zeta, w_r)$ either the partial derivative $F_\zeta(\zeta, w_r)$ or $F_w(\zeta, w_r)$ are non zero. It is easy to prove that the polynomial (54) is irreducible.

We can treat equation (54) as an algebraic relation which defines a Riemannian surface $\Gamma \ : \ F(\zeta, w) = 0$ of $w = w(\zeta)$ as a compact manifold over the ζ-sphere. The function $w(\zeta)$ is uniquely analytically extended over the Riemann surface Γ of two sheets over the $\zeta-$sphere. We find all singular or branch points of $w(\zeta)$ if we study the roots of the first coefficient of the polynomial $F(\zeta, w)$, the common roots of equations

$$F(\zeta, w) = 0, \quad F_w(\zeta, w) = 0, \qquad \zeta, w \in \mathbb{C} \cup \infty. \tag{55}$$

and the point $\zeta = \infty$. The set of singular or branch points consists of the points

$$\zeta_1 = 0, \ \ \zeta_2 = \frac{q}{4b}, \ \ \zeta_3 = \infty. \tag{56}$$

As expected we got the same set of points as in the real case (52) by the study of the Lipschitz condition but now the behavior of solutions at the points is more visible.

The points ζ_2, ζ_3 are the branch points at which two sheets of Γ are glued on. We remark that

$$w(\zeta_2) = -\frac{1}{b}\left(1 - \xi\frac{q}{4}\right) - t\frac{4\xi}{\sqrt{-bq}} + \cdots, \quad t^2 = \zeta - \frac{q}{4b}, \tag{57}$$

where t is a local parameter in the neighborhood of ζ_2. For the special value of $q = 4\xi$, i.e. $k = 0$, the value $w(\zeta_2)$ is equal to zero.

At the point $\zeta_3 = \infty$ we have

$$w(\zeta) = -\frac{\xi}{t^2} + \frac{1}{b} - \xi t\sqrt{\frac{-q}{4b^3}}, \quad t^2 = \frac{1}{\zeta}, \quad \zeta \to \infty,$$

where t is a local parameter in the neighborhood of ζ_3. At the point $\zeta_1 = 0$ the function $w(\zeta)$ has the following behavior

$$w(\zeta) \ \sim \ -\frac{q}{b^2}\frac{1}{\zeta}, \quad \zeta \to \zeta_1 = 0, \text{ on the principal sheet,} \tag{58}$$

$$w(\zeta) \ \sim \ -\left(\xi + \frac{1}{q}\right)\zeta, \quad \zeta \to \zeta_1 = 0, \ q \neq 1, \text{ on the second sheet,} \tag{59}$$

$$w(\zeta) \ \sim \ -2b^2\zeta^2, \quad \zeta \to \zeta_1 = 0, \ q = 1, \text{ on the second sheet.} \tag{60}$$

Any solution $w(\zeta)$ of an irreducible algebraic equation (54) is meromorphic on this compact Riemann surface Γ of the genus 0 and has a pole of the order one correspondingly (58) over the point $\zeta_1 = 0$ and the pole of the second order over $\zeta_3 = \infty$. It means also that the meromorphic function $w(\zeta)$ cannot be defined on a manifold of less than 2 sheets over the ζ sphere.

To solve the differential equations (50) and (51) from this point of view it is equivalent to integrate on Γ a differential of the type $\frac{\mathrm{d}\zeta}{w(\zeta)}$ and then to solve an Abel's inverse problem of degenerated type

$$\int \frac{\mathrm{d}\zeta}{w(\zeta)} = z + \mathrm{const.} \tag{61}$$

The integration can be done very easily because we can introduce a uniformizing parameter on the Riemann surface Γ and represent the integral (61) in terms of rational functions merged possibly with logarithmic terms.

To realize this program we introduce a new variable (our uniformizing parameter p) by

$$\zeta = \frac{q(1-p^2)}{4b}, \tag{62}$$

$$w = \frac{\xi(p-1)(q(1+p)^2+4\xi)}{4b(p+1)}. \tag{63}$$

Then the equations (50) and (51) will take the form

$$2q\xi \int \frac{p(p+1)\mathrm{d}p}{(p-1)(q(p+1)^2+4\xi)} = z + \text{const}, \tag{64}$$

$$2q\xi \int \frac{p(p-1)\mathrm{d}p}{(p+1)(q(p-1)^2+4\xi)} = z + \text{const}. \tag{65}$$

The integration procedure of equations (64), (65) gives rise to relations defining a complete set of first order differential equations. In order to see that these are first order ODEs recall that from the substitutions (53) and (43) we have

$$p = \sqrt{1 - \frac{4b}{q} v_z}. \tag{66}$$

We summarize all these results in the following theorem.

Theorem 3.3. *The equation (42) for arbitrary values of the parameters $q, b \neq 0$ can be reduced to the set of first order differential equations which consists of the equations*

$$v_z = 0, \quad v_z = (1 \pm 2)/b, \tag{67}$$

and equations (50), (51) where y is defined by the substitution (43). The complete set of solutions of the equation (42) coincides with the union of solutions of these equations.

To solve equations (64), (65) exactly we have first to integrate and then invert these formulas in order to obtain an exact representation of p as a function of z. If an exact formula for the function $p = p(z)$ is found we can use the substitution (66) to obtain an explicit ODE of the type $v_z(z) = f(z)$ or another suitable type; in that case it is possible to solve the problem completely. However, for an arbitrary value of the parameter q inversion is impossible, and we have just an implicit representation for the solutions of the equation (42) as solutions of the implicit first order differential equations.

3.4. Exact Invariant Solutions for a Fixed Relation between S and t

For a special value of the parameter q, namely for $q = -4$, we can integrate and invert the equations (64) and (65). For $q = -4$ the relation between the variables S and t is fixed in the form

$$z = \log S - \frac{\sigma^2}{8} t, \tag{68}$$

equation (64) takes the form

$$(p-1)^2(p+2) = 2\,c\,\exp\left(-3z/2\right), \tag{69}$$

and correspondingly the equation (65) becomes

$$(p+1)^2(p-2) = 2\,c\,\exp\left(-3z/2\right), \tag{70}$$

where c an arbitrary constant. It is easy to see that the equations (69) and (70) are connected by a transformation

$$p \to -p, \quad c \to -c. \tag{71}$$

This symmetry arises from the symmetry of the underlying Riemann surface Γ (54) and corresponds to a change of the sheets on Γ. Using these symmetry properties we can prove the following theorem.

Theorem 3.4. *The second order differential equation*

$$v_z - 4\frac{v_{zz}+v_z}{(1-b(v_{zz}+v_z))^2} = 0, \tag{72}$$

is exactly integrable for any value of the parameter b. The complete set of solutions for $b \neq 0$ is given by the union of solutions (76), (78)-(81) and solutions

$$v(z) = d, \quad v(z) = \frac{3}{b}z + d, \quad v(z) = -\frac{1}{b}z + d, \tag{73}$$

where d is an arbitrary constant. The last solution in (73) corresponds to the exceptional solution of equation (44).

For $b = 0$ equation (72) is linear and its solutions are given by $v(z) = d_1 + d_2\exp -(3z/4)$, where d_1, d_2 are arbitrary constants.

Proof. Because of the symmetry (71) it is sufficient to study either the equations (69) or (70) for $c \in \mathbb{R}$ or both these equations for $c > 0$. The value $c = 0$ can be excluded because it complies with the constant value of $p(z)$ and correspondingly constant value of $v_z(z)$, but all such cases are studied before and the solutions are given by (73).

We will study equation (70) in case $c \in \mathbb{R}\setminus\{0\}$ and obtain on this way the complete class of exact solutions for equations (69)-(70) and on this way for the equation (72).

Equation (70) for $c > 0$ has one real root only. It leads to an ODE of the form

$$\begin{aligned} v_z(z) = \frac{1}{b}\Bigg(1 &+ \left(1 + c\,e^{-\frac{3z}{2}} + \sqrt{2\,c\,e^{-\frac{3z}{2}} + c^2\,e^{-3z}}\right)^{-\frac{2}{3}} \\ &+ \left(1 + c\,e^{-\frac{3z}{2}} + \sqrt{2\,c\,e^{-\frac{3z}{2}} + c^2\,e^{-3z}}\right)^{\frac{2}{3}}\Bigg), \quad c > 0. \end{aligned} \tag{74}$$

Equation (74) can be exactly integrated if we use an Euler substitution and introduce a new independent variable

$$\tau = 2\left(1 + c\,e^{\frac{-3z}{2}} + \sqrt{2\,c\,e^{\frac{-3z}{2}} + c^2\,e^{-3z}}\right). \tag{75}$$

The corresponding solution is given by

$$
\begin{aligned}
v_r(z) = & -\frac{1}{b}\left(1 + c\, e^{-\frac{3\,z}{2}} + \sqrt{2\, c\, e^{-\frac{3\,z}{2}} + c^2\, e^{-3\,z}}\right)^{-\frac{2}{3}} \\
& -\frac{1}{b}\left(1 + c\, e^{-\frac{3\,z}{2}} + \sqrt{2\, c\, e^{-\frac{3\,z}{2}} + c^2\, e^{-3\,z}}\right)^{\frac{2}{3}} \\
& -\frac{2}{b}\log\left(\left(1 + c\, e^{-\frac{3\,z}{2}} + \sqrt{2\, c\, e^{-\frac{3\,z}{2}} + c^2\, e^{-3\,z}}\right)^{-\frac{1}{3}}\right. \\
& \left. + \left(1 + c\, e^{-\frac{3\,z}{2}} + \sqrt{2\, c\, e^{-\frac{3\,z}{2}} + c^2\, e^{-3\,z}}\right)^{\frac{1}{3}} - 2\right) + d\,,
\end{aligned} \tag{76}
$$

where $d \in \mathbb{R}$ is an arbitrary constant. If in the right hand side of equation (70) the parameter c satisfies the inequality $c < 0$ and the variable z chosen in the region

$$
z \in \left(-\frac{2}{3}\ln\frac{2}{|c|}, \infty\right) \tag{77}
$$

then the equation on p possesses maximal three real roots. These three roots of the cubic equation (70) give rise to three differential equations of the type $v_z = -(1 - p^2(z))/b$. These equations can be exactly solved and we find correspondingly three solutions $v_i(z),\ i = 1, 2, 3$.

The first solution is given by the expression

$$
\begin{aligned}
v_1(z) = & -\frac{z}{b} - \frac{2}{b}\,\cos\left(\frac{2}{3}\,\arccos\left(1 - |c|\, e^{\frac{-3\,z}{2}}\right)\right) \\
& -\frac{4}{3b}\,\log\left(1 + 2\,\cos\left(\frac{1}{3}\,\arccos\left(1 - |c|\, e^{\frac{-3\,z}{2}}\right)\right)\right) \\
& -\frac{16}{3b}\,\log\left(\sin\left(\frac{1}{6}\,\arccos\left(1 - |c|\, e^{\frac{-3\,z}{2}}\right)\right)\right) + d,
\end{aligned} \tag{78}
$$

where $d \in \mathbb{R}$ is an arbitrary constant. The second solution is given by the formula

$$
\begin{aligned}
v_2(z) = & -\frac{z}{b} - \frac{2}{b}\,\cos\left(\frac{2}{3}\pi - \frac{2}{3}\,\arccos\left(1 - |c|\, e^{\frac{-3\,z}{2}}\right)\right) \\
& -\frac{4}{3b}\,\log\left(-1 + 2\,\cos\left(\frac{1}{3}\pi - \frac{1}{3}\,\arccos\left(1 - |c|\, e^{\frac{-3\,z}{2}}\right)\right)\right) \\
& -\frac{16}{3b}\,\log\left(\sin\left(\frac{1}{6}\pi - \frac{1}{6}\,\arccos\left(1 - |c|\, e^{\frac{-3\,z}{2}}\right)\right)\right) + d,
\end{aligned} \tag{79}
$$

where $d \in \mathbb{R}$ is an arbitrary constant. The first and second solutions are defined up to the point $z = -\frac{2}{3}\ln\frac{2}{|c|}$ where they coincide (see Fig. 1).

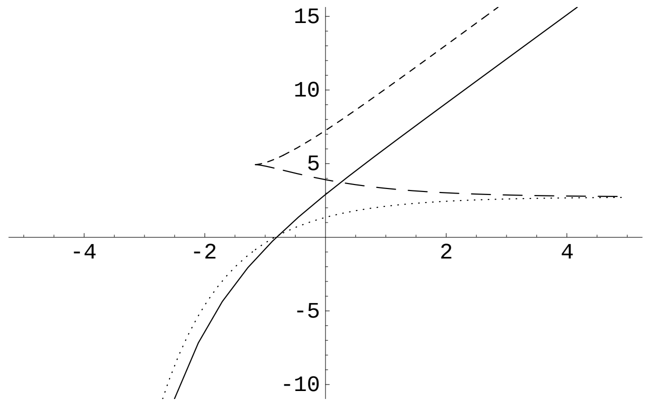

Figure 1. Plot of the solution $v(z)$ given in (76) (thick solid line); $v_1(z)$ from (78) (short dashed line); $v_2(z)$ from (79) (long dashed line) and the third solution $v_{3,1}(z), v_{3,2}(z)$ from (80), (81) (thin solid line). The parameters takes the values $|c| = 0.35$, $q = -4$, $d = 0$, $b = 1$ and the variable $z \in (-5, 4.5)$.

The third solution for $z > -\frac{2}{3} \ln \frac{2}{|c|}$ is given by the formula

$$\begin{aligned} v_{3,2}(z) = &-\frac{z}{b} - \frac{2}{b} \cos\left(\frac{2}{3}\pi + \frac{2}{3} \arccos\left(1 - |c|\, e^{\frac{-3\,z}{2}}\right)\right) \\ &- \frac{4}{3b} \log\left(-1 + 2 \cos\left(\frac{1}{3}\pi + \frac{1}{3} \arccos\left(-1 + |c|\, e^{\frac{-3\,z}{2}}\right)\right)\right) \\ &- \frac{16}{3b} \log\left(\cos\left(\frac{1}{6}\pi + \frac{1}{6} \arccos\left(-1 + |c|\, e^{\frac{-3\,z}{2}}\right)\right)\right) + d, \end{aligned} \tag{80}$$

where $d \in \mathbb{R}$ is an arbitrary constant. In the case $z < \frac{2}{3} \ln \frac{2}{|c|}$ the polynomial (70) has one real root and the corresponding solution can be represented by the formula

$$\begin{aligned} v_{3,1}(z) = &-\frac{z}{b} - \frac{2}{b} \cosh\left(\frac{2}{3}\operatorname{arccosh}\left(-1 + |c|\, e^{\frac{-3\,z}{2}}\right)\right) \\ &- \frac{16}{3\,b} \log\left(\cosh\left(\frac{1}{6}\operatorname{arccosh}\left(-1 + |c|\, e^{\frac{-3\,z}{2}}\right)\right)\right) \\ &- \frac{4}{3\,b} \log\left(-1 + 2 \cosh\left(\frac{1}{3}\operatorname{arccosh}\left(-1 + |c|\, e^{\frac{-3\,z}{2}}\right)\right)\right) + d. \end{aligned} \tag{81}$$

The third solution is represented by formulas $v_{3,2}(z)$ and $v_{3,1}(z)$ for different values of the variable z.

One of the sets of solutions (76), (78) -(81) for fixed parameters b, c, d is represented in Fig. 1. The first solution (74) and the third solution given by both (80) and (81) are defined for any values of z. The solutions $v_1(z)$ and $v_2(z)$ cannot be continued to the left after the point $z = -\frac{2}{3} \ln \frac{2}{|c|}$ where they coincide.

If we keep in mind that $z = \log S - \frac{\sigma^2}{8} t$ and $u(S,t) = S\ v(z)$ we can represent the complete set of exact invariant solution of equation (9). The solution (76) gives rise to an

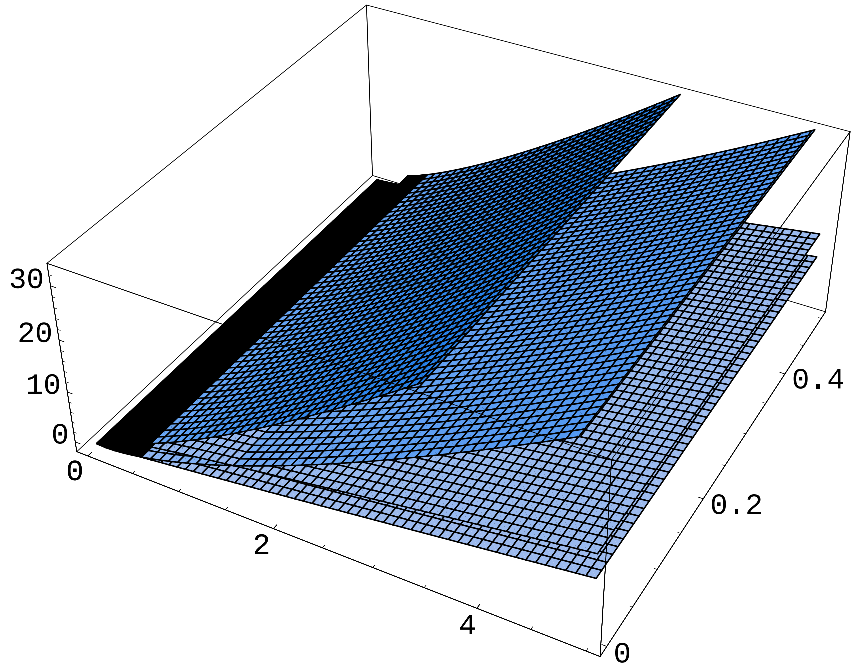

Figure 2. Plot of solutions $u_r(S,t), u_1(S,t), u_2(S,t)$ and $u_{3,1}(S,t), u_{3,2}(S,t)$ for the parameters $\sigma = 0.4,\ |c| = 0.5, q = -4, b = 1.0, d = 0$. The variables S, t lie in intervals $S \in (0., 5.)$ and $t \in [0, 0.5]$. All invariant solutions change slowly in t-direction.

invariant solution $u_r(S,t)$ in the form

$$\begin{aligned} u_r(S,t) &= -\frac{1}{\omega\rho} S\left(1 + c\,S^{-\frac{3}{2}} e^{\frac{3\sigma^2}{16}t} + \sqrt{2\,c\,S^{-\frac{3}{2}} e^{\frac{3\sigma^2}{16}t} + c^2\,S^{-3} e^{\frac{3\sigma^2}{8}t}}\right)^{-\frac{2}{3}} \\ &- \frac{1}{\omega\rho} S\left(1 + c\,S^{-\frac{3}{2}} e^{\frac{3\sigma^2}{16}t} + \sqrt{2\,c\,S^{-\frac{3}{2}} e^{\frac{3\sigma^2}{16}t} + c^2\,S^{-3} e^{\frac{3\sigma^2}{8}t}}\right)^{\frac{2}{3}} \\ &- \frac{2}{\omega\rho} S \log\left(\left(1 + c\,S^{-\frac{3}{2}} e^{\frac{3\sigma^2}{16}t} + \sqrt{2\,c\,S^{-\frac{3}{2}} e^{\frac{3\sigma^2}{16}t} + c^2\,S^{-3} e^{\frac{3\sigma^2}{8}t}}\right)^{-\frac{1}{3}}\right. \\ &+ \left.\left(1 + c\,S^{-\frac{3}{2}} e^{\frac{3\sigma^2}{16}t} + \sqrt{2\,c\,S^{-\frac{3}{2}} e^{\frac{3\sigma^2}{16}t} + c^2\,S^{-3} e^{\frac{3\sigma^2}{8}t}}\right)^{\frac{1}{3}} - 2\right) + d\,S + d_2 \end{aligned} \tag{82}$$

where $d,\ d_2 \in \mathbb{R}$, $c > 0$. This solution was obtained and studied in [6]. We describe now other invariant solutions from the complete set of invariant solutions.

In the case $c < 0$ we can obtain correspondingly three real solutions if

$$S \geq \left(\frac{|c|}{2}\right)^{2/3} \exp\left(\frac{\sigma^2}{8}t\right). \tag{83}$$

The first solution is represented by

$$
\begin{aligned}
u_1(S,t) = &-\frac{1}{\omega\rho}S\left(\log S-\frac{\sigma^2}{8}t\right)-\frac{2}{\omega\rho}\,S\cos\left(\frac{2}{3}\,\arccos\left(1-|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\\
&-\frac{4}{3\omega\rho}\,S\log\left(1+2\,\cos\left(\frac{1}{3}\,\arccos\left(1-|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\right)\\
&-\frac{16}{3\omega\rho}\,S\log\left(\sin\left(\frac{1}{6}\,\arccos\left(1-|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\right)+d\,S+d_2,
\end{aligned}
\tag{84}
$$

where $d,\ d_2\in\mathbb{R}$, $c<0$. The second solution is given by the formula

$$
\begin{aligned}
u_2(S,t) \;=\; &-\frac{1}{\omega\rho}S\left(\log S-\frac{\sigma^2}{8}t\right)-\frac{2}{\omega\rho}\,S\cos\left(\frac{2}{3}\pi+\frac{2}{3}\,\arccos\left(-1+|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\\
&-\frac{4}{3\omega\rho}\,S\log\left(1+2\,\cos\left(\frac{1}{3}\pi+\frac{1}{3}\,\arccos\left(-1+|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\right)\\
&-\frac{16}{3\omega\rho}\,S\log\left(\sin\left(\frac{1}{6}\pi+\frac{1}{6}\,\arccos\left(-1+|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\right)+d\,S+d_2.
\end{aligned}
\tag{85}
$$

where d , $d_2\in\mathbb{R}$, $c<0$. The first and second solutions are defined for the variables under conditions (83). They coincide along the curve

$$
S=\left(\frac{|c|}{2}\right)^{2/3}\exp\left(\frac{\sigma^2}{8}t\right)
$$

and cannot be continued further. The third solution is defined by

$$
\begin{aligned}
u_{3,2}(S,t) \;=\; &-\frac{1}{\omega\rho}S\left(\log S-\frac{\sigma^2}{8}t\right)\\
&-\frac{2}{\omega\rho}\,S\cos\left(\frac{2}{3}\,\arccos\left(-1+|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\\
&-\frac{4}{3\omega\rho}\,S\log\left(-1+2\,\cos\left(\frac{1}{3}\,\arccos\left(-1+|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\right)\\
&-\frac{16}{3\omega\rho}\,S\log\left(\cos\left(\frac{1}{6}\,\arccos\left(-1+|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\right)+d\,S+d_2,
\end{aligned}
\tag{86}
$$

where d, $d_2\in\mathbb{R}$ and S,t satisfy the condition (83).

In case $0<S<\left(\frac{|c|}{2}\right)^{\frac{2}{3}}\exp\left(\frac{\sigma^2}{8}t\right)$ the third solution can be represented by the formula

$$
\begin{aligned}
u_{3,1}(S,t) \;=\; &-\frac{1}{\omega\rho}S\left(\log S-\frac{\sigma^2}{8}t\right)\\
&-\frac{2}{\omega\rho}\,S\cosh\left(\frac{2}{3}\operatorname{arccosh}\left(-1+|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\\
&-\frac{16}{3\,\omega\rho}\,S\log\left(\cosh\left(\frac{1}{6}\operatorname{arccosh}\left(-1+|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\right)\\
&-\frac{4}{3\,\omega\rho}\,S\log\left(-1+2\,\cosh\left(\frac{1}{3}\operatorname{arccosh}\left(-1+|c|\,S^{-\frac{3}{2}}e^{\frac{3\sigma^2}{16}t}\right)\right)\right)\\
&+\;d\,S+d_2,
\end{aligned}
\tag{87}
$$

where $d,\ d_2 \in \mathbb{R}$. The solution $u(S,t)$ (82) and the third solution given by $u_{3,1}$, $u_{3,2}$ (86), (87) are defined for all values of variables t and $S > 0$. They have a common intersection curve of the type $S = \text{const.}\ \exp(\frac{\sigma^2}{8}t)$. The typical behavior of all these invariant solutions is represented in Fig. 2.

We summarize the previous results in the following theorem which describes the set of invariant solutions of equation (9).

Theorem 3.5. *The invariant solutions of equation (9) can be defined by the set of first order ordinary differential equations listed in Theorem 3.3.*

If moreover the parameter $q = -4$, or equivalent in the substitution (41) we chose $a = -\sigma^2/8$ then the complete set of invariant solutions of (9) can be found exactly. This set of invariant solutions is given by formulas (82)–(87) and by solutions

$$u(S,t) = d\,S, \quad u(S,t) = \frac{3}{b}\,S\,(\log S - \sigma^2 \frac{t}{8}),$$
$$u(S,t) = -\frac{1}{b}\,S\,(\log S - \sigma^2 \frac{t}{8}),$$

where d denotes an arbitrary constant. This set of invariant solutions is unique up to the transformations of the symmetry group G_Δ given by Theorem 3.2.

The solutions $u_r(S,t)$, (82), $u_1(S,t)$, (84), $u_2(S,t)$, (85), $u_{3,1}(S,t)$, (86), $u_{3,2}(S,t)$ (87), have no one counterpart in a linear case. If in the equation (9) the parameter $\rho = 0$ the equation becomes linear. If the parameter $\rho \to 0$, then equation (9) and correspondingly equation (37) will be reduced to the linear Black-Scholes equation but the solutions (82)-(87) which we obtained here will be completely blown up by $\rho \to 0$ because of the factor $1/b = 1/(\omega\rho)$ in the formulas (82)-(87). This phenomena was described as well in [6], [9], where the solution $u_r(S,t)$ was studied and for the complete set of invariant solutions of equation (37) with $k = 1$ in [5].

4. Properties of Solutions and Parameter-Sensitivity

We study the properties of solutions, keeping in mind that because of the symmetry properties (see Theorem 3.2) of the equation (9) we can add to each solution a linear function of the variable S.

4.1. Dependence on the Constant c and Terminal Payoff

First we study the dependence of the solution u_2 on the arbitrary constant c. The constant c is the first constant of integration of the ODEs (50), (51). This dependence is illustrated in Figure 3. We see on Fig. 3 that for different values of the constant c the domains of the solutions are different. The dependence of the solutions $u_{3,1}, u_{3,2}(S,t)$ on the constant c is exemplified in Figure 4.

We obtain a typical terminal payoff function for the solutions (82)-(87) if we just fix $t = T$. By changing the value of the constant c and by adding a linear function of S we are able to modify terminal payoff function for the solutions. Hence we can approximate typical payoff profiles of financial derivatives quite well.

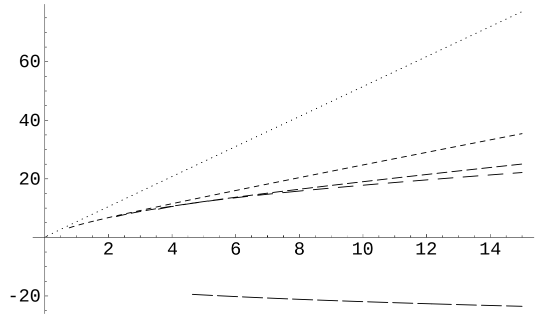

Figure 3. Plot of solutions $u_2(S,t)$ where $|c| = 0.01, 1., 5., 10., 20., \sigma = 0.4, q = -4$, $b = 1.0, d = 0$. The variables S lie in intervals $S \in (0., 15.)$ and $t = 1.$.

The curves going from up to down with the growing value of the parameter $|c|$.

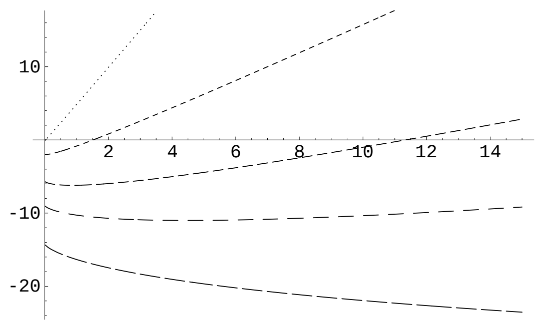

Figure 4. Plot of solutions $u_3(S,t)$ where $|c| = 0.01, 1., 5., 10., 20., \sigma = 0.4, q = -4$, $b = 1.0, d = 0$. The variables S lie in intervals $S \in (0., 15.)$ and $t = 1.$.

The curves going from up to down with the growing value of the parameter $|c|$.

4.2. Dependence on Time

All solutions depend weakly on time because of the substitution (68) all invariant solutions depends on the combination $\sigma^2 t$. As long as we take the volatility σ to be small we obtain a dampened dependence of the solutions on time.

In Fig. 5 we can see this dependence for examples of the solution $u_{3,1}(S,t), u_{3,2}(S,t)$.

4.3. Dependence on the Parameter ρ

All solutions found in this work have the form

$$u(S,t) = w(S,t)/\rho, \tag{88}$$

where $w(S,t)$ is a smooth function of S, t. Hence the function $w(S,t)$ solves the equation (9) with $\rho = 1$,

$$w_t + \frac{\sigma^2 S^2}{2} \frac{w_{SS}}{(1 - S w_{SS})^2} = 0. \tag{89}$$

Because of this relation any ρ-dependence of invariant solutions of (9) is trivial. In particular, if the terminal conditions are fixed, $u(S,T) = h(S)$, then the value $u(S,t)$ will increase

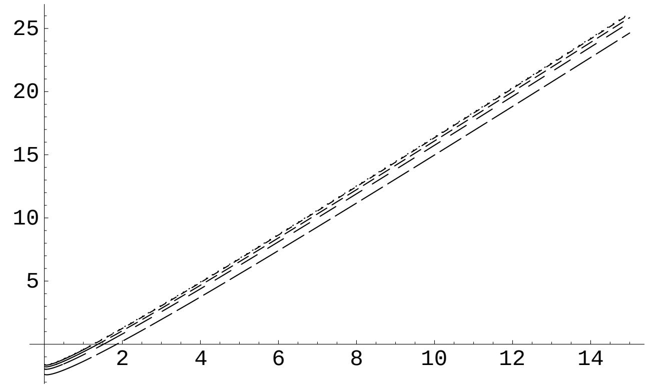

Figure 5. Plot of solutions $u_{3,1}(S,t), u_{3,2}(S,t)$ for the parameters $\sigma = 0.4$, $|c| = 10., q = -4, b = 1.0, d = 0$. The variables S lie in intervals $S \in (0., 15.)$ and $t = 0.01, 1., 5., 10., 20.$. The highest level corresponds to $t = 0.01$ and the lowest to $t = 20.$.

if the value of the parameter ρ increases. This dependence of hedge costs on the position of the large trader on the market is very natural.

4.4. Dependence on the Asset Price S

In practice one use often delta-hedging to reduce the sensitivity of a portfolio to the movements of an underlying asset. Hence it is important to know the value Δ defined by $\Delta = \frac{\partial u}{\partial S}$, where u denotes the value of the derivative product or of a portfolio. Using the exact formulas for the invariant solutions we can easily calculate Δ as a function of S and t. The Δ corresponding to the solution u_2 is presented in Fig. 6

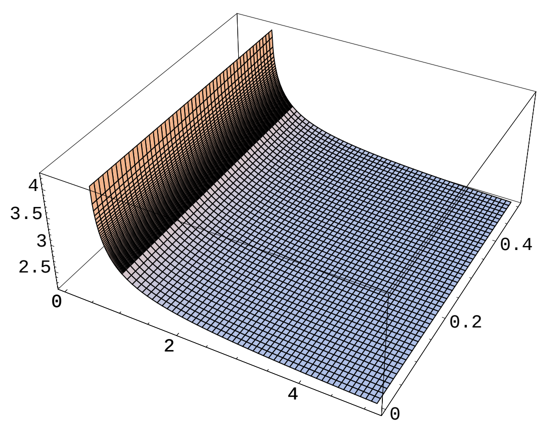

Figure 6. Plot of Δ for $u_2(S,t)$ and $\sigma = 0.4$, $|c| = 1., q = -4, b = 1.0, d = 0$. The variables lie in intervals $S \in (0., 5.)$ and $t = [0., 0.5]$.

The Δ of the solution $u_{3,1}, u_{3,2}$ is represented on the Fig. 7.

We see in both cases the strong dependence on S for small value of S. If $S \to \infty$, in both cases the Δ tends to a constant which is independent of time and the constant c.

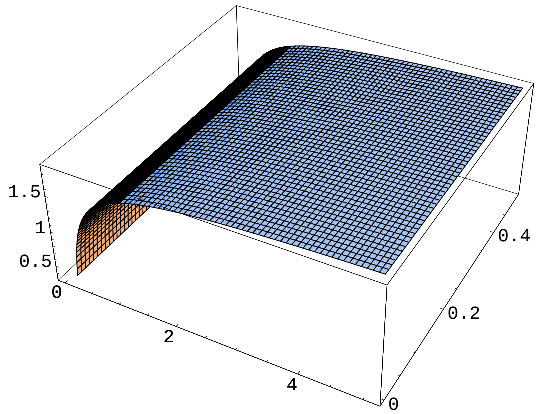

Figure 7. Plot of the Δ for $u_{3,1}(S,t), u_{3,2}(S,t)$ and $\sigma = 0.4,\ |c| = 1., q = -4, b = 1.0, d = 0$. The variables lie in intervals $S \in (0., 5.)$ and $t \in [0.01, 0.5]$.

4.5. Asymptotic Behavior of Invariant Solutions

If S is large enough we have four well defined solutions $u_r(S,t)$, $u_1(S,t)$, $u_2(S,t)$, $u_{3,2}(S,t)$. The asymptotic behavior solutions $u_r(S,t)$ from (82) and u_1 from (84) coincides in the main terms as $S \to \infty$ and is given by the formula

$$u_r(S,t),\ u_1(S,t) \sim \frac{1}{b}\left(3S\ln S + \text{const.}S + \mathcal{O}(S^{-1/2})\right). \tag{90}$$

The exact formula for the asymptotic behavior of the function $u_1(S,t)$ for $S \to \infty$ is given by

$$\begin{aligned} u_1(S,t) \quad &\sim \quad \frac{3}{b}\,S\log(S) + \frac{1}{b}S\left(4\,\log(3) - 2 - \frac{8}{3}\,\log\left(\frac{2}{|c|}\right) - \frac{3}{8}\sigma^2\,t\right) \\ &+ \quad \frac{2^4}{3^3 b}\,|c|\,e^{\frac{3\,\sigma^2\,t}{16}}\,S^{-\frac{1}{2}} + \mathcal{O}\left(S^{-\frac{5}{4}}\right). \end{aligned} \tag{91}$$

We see that for $S \to \infty$, the main term does not depend on time or on the value of the constant c. Moreover, this term cannot be changed by adding a linear function of S to the solution.

The main terms of the solutions $u_2(S,t)$ from (85) and $u_{3,2}$ from (87) behave similar to each other as $S \to \infty$; this behavior is given by the formulas

$$\begin{aligned} u_2(S,t) \quad &\sim \quad \frac{1}{b}\left(1 + \frac{2}{3}\,\log\left(\frac{2^7}{3^3|c|}\right)\right)\,S \\ &+ \quad \frac{2^3}{3b}\sqrt{\frac{2|c|}{3}}\,e^{\frac{3\,\sigma^2\,t}{32}}\,S^{\frac{1}{4}} - \frac{2^3}{3^3 b}\,|c|\,e^{\frac{3\,\sigma^2\,t}{16}}\,S^{-\frac{1}{2}} + \mathcal{O}\left(S^{-\frac{5}{4}}\right) \end{aligned} \tag{92}$$

and

$$u_{3,2}(S,t) \quad \sim \quad \frac{1}{b}\left(1 + \frac{2}{3}\,\log\left(\frac{2^7}{3^3|c|}\right)\right)\,S \tag{93}$$

$$- \quad \frac{2^3}{3b}\sqrt{\frac{2|c|}{3}}\,e^{\frac{3\,\sigma^2\,t}{32}}\,S^{\frac{1}{4}} - \frac{2^3}{3^3 b}\,|c|\,e^{\frac{3\,\sigma^2\,t}{16}}\,S^{-\frac{1}{2}} + \mathcal{O}\left(S^{-\frac{5}{4}}\right). \tag{94}$$

Note that the main term in formulas (92) and (93) depends on S linearly and has a coefficient which depends on the constant c only. Hence we can change the asymptotic behavior of the solutions $u_2(S,t)$ and $u_{3,1}(S,t)$, $u_{3,2}(S,t)$ by simply adding a linear function of S to the solutions.

For S in a neighborhood of $S \to 0$ there exist just two non trivial real invariant solutions of equation (9), i.e. the known solution $u_r(S,1)$ and the new solution $u_{3,1}(S,t)$. Using the exact formulas for the last solution we retain the first term and obtain as $S \to 0$ for the solution $u_{3,1}(S,t)$

$$u_{3,1}(S,t) \sim \frac{-14}{b}\, S \ln(S) + \mathcal{O}(S)\,. \tag{95}$$

The main term in this formula do not depend on t or constant c and can be changed by adding a linear function of S to the solution.

Conclusion

In this chapter we obtained explicit solutions for the equation (9) and studied their analytical properties. These solutions are useful for a number of reasons. To begin with, while the payoffs of these similarity solutions cannot be chosen arbitrary, the payoffs can be modified using embedded constants to tailor a given portfolio reasonably well. For some values of the parameters c, d we obtain a payoff typical for futures, for other values c, d the payoff is very similar to the form of calls (see Figure 3 and Figure 4). Moreover, the explicit solutions can be used as benchmark for different numerical methods.

An important difference between the case of the linear Black-Scholes equation and these nonlinear cases can be noticed if we consider the asymptotic for $S \to \infty$. In the linear case the price of a Call option satisfies $u(S,t) \to \mathrm{const} \cdot S$. In the nonlinear case the families of similarity solutions u_2 and u_3 which approximate the payoff of a Call option well on a finite interval $[0, \bar{S}]$, grow faster than linear as $S \to \infty$; see the formulas (92) and (93) for details. This reflects the fact that in illiquid markets option hedging is more expensive than in the standard case of perfectly liquid markets.

Invariant solutions can be used as benchmarks for different numerical methods or as a starting point for stability investigations of numerical schemes. We refer to [6] for further information on this issue.

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